Digitized by the Internet Archive
in 2022 with funding from
Kahle/Austin Foundation

https://archive.org/details/elementarytreati0000edwa

MATHEMATICAL WORKS,

By PROFESSOR EDWARD A. BOWSER.

ACADEMIC ALGEBRA. With Numerous Examples.

COLLEGE ALGEBRA. With Numerous Examples.

PLANE AND SOLID GEOMETRY. With Numerous Exercises.

ELEMENTS OF PLANE AND SPHERICAL TRIGONOMETRY. With Numerous Examples.

A TREATISE ON PLANE AND SPHERICAL TRIGONOMETRY, and its Applications to Astronomy and Geodesy. With Numerous Examples.

AN ELEMENTARY TREATISE ON ANALYTIC GEOMETRY, Embracing Plane Geometry and an Introduction to Geometry of Three Dimensions. With Numerous Examples.

AN ELEMENTARY TREATISE ON THE DIFFERENTIAL AND INTEGRAL CALCULUS. With Numerous Examples.

AN ELEMENTARY TREATISE ON ANALYTIC MECHANICS. With Numerous Examples.

AN ELEMENTARY TREATISE ON HYDROMECHANICS. With Numerous Examples.

LOGARITHMIC AND TRIGONOMETRIC TABLES.

A TREATISE ON ROOFS AND BRIDGES. With Numerous Exercises.

AN

ELEMENTARY TREATISE

ON

ANALYTIC MECHANICS

WITH NUMEROUS EXAMPLES.

BY

EDWARD A. BOWSER, LL.D.,

PROFESSOR OF MATHEMATICS AND ENGINEERING IN RUTGERS COLLEGE.

SEVENTEENTH EDITION.

NEW YORK:
D. VAN NOSTRAND COMPANY,
23 Murray and 27 Warren Sts.,
1904.

QA 805
.B78
1904

COPYRIGHT, 1884, BY E. A. BOWSER.

PREFACE.

THE present work on Analytic Mechanics or Dynamics is designed as a text-book for the students of Scientific Schools and Colleges, who have received training in the elements of Analytic Geometry and the Calculus.

Dynamics is here used in its true sense as the science of *force*. The tendency among the best and most logical writers of the present day appears to be to use this term for the science of Analytic Mechanics, while the branch formerly called Dynamics is now termed Kinetics.

The treatise is intended especially for beginners in this branch of science. It involves the use of Analytic Geometry and the Calculus. The analytic method has been chiefly adhered to, as being better adapted to the treatment of the subject, more general in its application and more fruitful in results than the geometric method; and yet where a geometric proof seemed preferable it has been introduced.

The aim has been to make every principle clear and intelligible, to develop the different theories with simplicity, and to explain fully the meaning and use of the various analytic expressions in which the principles are embodied.

The book consists of three parts. Part I, with the exception of a preliminary chapter devoted to definitions and fundamental principles, is entirely given to *Statics*.

Part II is occupied with Kinematics, and the principles of this important branch of mathematics are so treated that the student may enter upon the study of Kinetics with clear notions of motion, velocity and acceleration. Part III treats of the Kinetics of a particle and of rigid bodies

GETTYSBURG COLLEGE
50696
Gettysburg, Pa.

In this arrangement of the work, with the exception of Kinematics, I have followed the plan usually adopted, and made the subject of Statics precede that of Kinetics.

For the attainment of that grasp of principles which it is the special aim of the book to impart, numerous examples are given at the ends of the chapters. The greater part of them will present no serious difficulty to the student, while a few may tax his best efforts.

In preparing this book I have availed myself of the writings of many of the best authors. The chief sources from which I have derived assistance are the treatises of Price, Minchin, Todhunter, Pratt, Routh, Thomson and Tait, Tait and Steele, Weisbach, Venturoli, Wilson, Browne, Gregory, Rankine, Boucharlat, Pirie, Lagrange, and La Place, while many valuable hints as well as examples have been obtained from the works of Smith, Wood, Bartlett, Young, Moseley, Tate, Magnus, Goodeve, Parkinson, Olmsted, Garnett, Renwick, Bottomley, Morin, Twisden, Whewell, Galbraith, Ball, Dana, Byrne, the Encyclopædia Britannica, and the Mathematical Visitor.

I have again to thank my old pupil, Mr. R. W. Prentiss, of the Nautical Almanac Office, and formerly Fellow in Mathematics at the Johns Hopkins University, for reading the MS. and for valuable suggestions. Several others also of my friends have kindly assisted me by correcting proof-sheets and verifying copy and formulæ.

E. A. B.

RUTGERS COLLEGE,
NEW BRUNSWICK, N. J., *June*, 1884.

TABLE OF CONTENTS.

PART I.

CHAPTER I.

FIRST PRINCIPLES.

ART. PAGE
1. Definitions—Statics, Kinetics and Kinematics.............. 1
2. Matter.. 2
3. Inertia... 2
4. Body, Space and Time................................ 2
5. Rest and Motion..................................... 2
6. Velocity.. 3
7. Formulæ for Velocity................................ 4
8. Acceleration.. 4
9. Measure of Acceleration............................. 5
10. Geometric Representation of Velocity and Acceleration...... 6
11. Mass... 6
12. Momentum... 7
13. Change of Momentum................................. 7
14. Force.. 8
15. Static Measure of Force............................ 8
16. Action and Reaction................................ 9
17. Method of Comparing Forces......................... 9
18. Representation of Forces........................... 10
19. Measure of Accelerating Forces..................... 10
20. Kinetic Measure of Force........................... 11
21. Absolute or Kinetic Unit of Force.................. 13
22. Three Ways of Measuring Force...................... 14
23. Meaning of *g* in Dynamics......................... 15

ART. PAGE
24. Gravitation Units of Force and Mass........................ 16
25. Gravitation and Absolute Measure........................ 17
Examples 18

STATICS (REST).

CHAPTER II.

THE COMPOSITION AND RESOLUTION OF CONCURRING FORCES—CONDITIONS OF EQUILIBRIUM.

26. Problem of Statics.. 21
27. Concurring and Conspiring Forces........................ 21
28. Composition of Conspiring Forces........................ 22
29. Composition of Velocities................................ 23
30. Composition of Forces.................................... 24
31. Triangle of Forces.. 25
32. Three Concurring Forces in Equilibrium.................. 26
33. The Polygon of Forces.................................... 27
34. Parallelopiped of Forces.................................. 28
35. Resolution of Forces...................................... 30
36. Magnitude and Direction of Resultant.................... 31
37. Conditions of Equilibrium................................ 32
38. Resultant of Concurring Forces in Space.................. 34
39. Equilibrium of Concurring Forces in Space................ 35
40. Tension of a String...................................... 35
41. Equilibrium of Concurring Forces on a Smooth Plane....... 39
42. Equilibrium of Concurring Forces on a Smooth Surface..... 41
Examples 45

CHAPTER III.

COMPOSITION AND RESOLUTION OF FORCES ACTING ON A RIGID BODY.

43. A Rigid Body.. 57
44. Transmissibility of Force................................ 57
45. Resultant of Two Parallel Forces........................ 58
46. Moment of a Force 60

ART. PAGE
47. Signs of Moments.... 61
48. Geometric Representation of a Moment.... 61
49. Two Equal and Opposite Parallel Forces.... 61
50. Moment of a Couple.... 62
51. Effect of a Couple on a Rigid Body.... 63
52. Effect of Transferring Couple to Parallel Plane not altered... 64
53 A Couple replaced by another Couple.... 64
54. A Force and a Couple.... 65
55. Resultant of any number of Couples.... 66
56. Resultant of Two Couples.... 67
57. Varignon's Theorem of Moments.... 69
58. Varignon's Theorem for Parallel Forces 71
59. Centre of Parallel Forces.... 71
60. Equilibrium of a Rigid Body under Parallel Forces.... 74
61. Equilibrium of a Rigid Body under Forces in any Direction.. 75
62. Equilibrium under Three Forces.... 77
63. Centre of Parallel Forces in Different Planes.... 85
64. Equilibrium of Parallel Forces in Space.... 86
65. Equilibrium of Forces acting in any Direction in Space.... 88
Examples.... 90

CHAPTER IV.

CENTRE OF GRAVITY (CENTRE OF MASS).

66. Centre of Gravity.... 100
67. Planes of Symmetry—Axes of Symmetry.... 101
68. Body Suspended from a Point.... 101
69. Body Supported on a Surface.... 102
70. Different Kinds of Equilibrium.... 102
71. Centre of Gravity of Two Masses.... 103
72. Centre of Gravity of Part of a Body.... 103
73. Centre of Gravity of a Triangle.... 103
74. Centre of Gravity of a Triangular Pyramid.... 105
75. Centre of Gravity of a Cone.... 106
76. Centre of Gravity of Frustum of Pyramid.... 107
77. Investigations involving Integration.... 109
78. Centre of Gravity of the Arc of a Curve.... 110
79. Centre of Gravity of a Plane Area.... 115
80. Polar Elements of a Plane Area.... 118

ART. PAGE.
81. Double Integration—Polar Formulæ........................ 120
82. Double Integration—Rectangular Formulæ................ 122
83. Centre of Gravity of a Surface of Revolution..... 123
84. Centre of Gravity of any Curved Surface.................. 126
85. Centre of Gravity of a Solid of Revolution................. 127
86. Polar Formulæ... 130
87. Centre of Gravity of any Solid............................ 131
88. Polar Elements of Mass.................................... 133
89. Special Methods... 136
90. Theorems of Pappus.. 138
Examples.. 140

CHAPTER V.

FRICTION.

91. Friction............ 149
92. Laws of Friction... 150
93. Magnitudes of Coefficients of Friction............ 152
94. Angle of Friction.. 152
95. Reaction of a Rough Curve or Surface..................... 153
96. Friction on an Inclined Plane............................ 154
97. Friction on a Double Inclined Plane...................... 156
98. Friction on Two Inclined Planes.......................... 159
99. Friction of a Trunnion................................... 159
100. Friction of a Pivot....................................... 160
Examples 162

CHAPTER VI.

THE PRINCIPLE OF VIRTUAL VELOCITIES.

101. Virtual Velocity.. 166
102. Principle of Virtual Velocities........................... 167
103. Nature of the Displacement..... 169
104. Equation of Virtual Moments............................. 169
105. System of Particles Rigidly Connected.................... 170
Examples 172

CHAPTER VII.

MACHINES.

ART. PAGE
106. Functions of a Machine.................................. 177
107. Mechanical Advantage.................................. 178
108. Simple Machines.................................. 180
109. The Lever.................................. 181
110. Equilibrium of the Lever.................................. 181
111. The Common Balance.................................. 184
112. Chief Requisites of a Good Balance.................................. 186
113. The Steelyard.................................. 188
114. To Graduate the Common Steelyard.................................. 188
115. The Wheel and Axle.................................. 190
116. Equilibrium of the Wheel and Axle.................................. 190
117. Toothed Wheels.................................. 192
118. Relation of Power and Weight in Toothed Wheels.................................. 193
119. Relation of Power to Weight in a Train of Wheels.................................. 194
120. The Inclined Plane.................................. 196
121. The Pulley.................................. 197
122. The Simple Movable Pulley.................................. 198
123. First System of Pulleys.................................. 198
124. Second System of Pulleys.................................. 200
125. Third System of Pulleys.................................. 201
126. The Wedge.................................. 202
127. Mechanical Advantage of the Wedge.................................. 202
128. The Screw.................................. 204
129. Relation between Power and Weight in the Screw.................................. 204
129*a*. Prony's Differential Screw.................................. 206
Examples.................................. 207

CHAPTER VIII.

THE FUNICULAR POLYGON—THE CATENARY—ATTRACTION

130. Equilibrium of the Funicular Polygon.................................. 216
131. To Construct the Funicular Polygon.................................. 218
132. Cord Supporting a Load Uniformly Distributed.................................. 219
133. The Common Catenary—Its Equation.................................. 221
133*a*. Attraction of a Spherical Shell.................................. 226
Examples.................................. 228

PART II.

KINEMATICS (MOTION).

CHAPTER I.

RECTILINEAR MOTION.

ART. PAGE
134. Definitions—Velocity 231
135. Acceleration 233
136. Relation between Space and Time when Acceleration = 0... 233
137. Relation when the Acceleration is Constant 234
138. Relation when Acceleration varies as the Time 235
139. Relation when Acceleration Varies as the Distance 235
140. Equations of Motion for Falling Bodies 237
141. Particle Projected Vertically Upwards 239
142. Compositions of Velocities 242
143. Resolution of Velocities 243
144. Motion on an Inclined Plane 245
145. Times of Descent down Chords of a Circle 247
146. The Straight Line of Quickest Descent 248
Examples 249

CHAPTER II.

CURVILINEAR MOTION.

147. Remarks on Curvilinear Motion 258
148. Composition of Uniform Velocity and Acceleration 258
149. Composition and Resolution of Acceleration 259
Examples 261
150. Motion of Projectiles in Vacuo 266
151. The Path of a Particle in Vacuo is a Parabola 266
152. The Parameter—Range—Greatest Height—Height of Directrix 267
153. Velocity of a Particle at any point of its Path 269
154. Time of Flight along a Horizontal Plane 269
155. Point at which a Projectile will Strike an Inclined Plane... 270

ART. PAGE
156. Projection for Greatest Range on a Given Plane............ 270
157. The Elevation that the Particle may pass a Given Point... 271
158. Second Method of Finding Equation of Trajectory......... 272
159. Velocity of Discharge of Balls and Shells.................. 274
160. Angular Velocity and Angular Acceleration................ 275
161. Accelerations Along and Perpendicular to Radius Vector.... 278
162. Accelerations Along and Perpendicular to Tangent.... 279
163. When Acceleration Perpendicular to Radius Vector is zero.. 281
164. When Angular Velocity is Constant........................ 282
Examples.. 284

PART III.

KINETICS (MOTION AND FORCE).

CHAPTER I.

LAWS OF MOTION—MOTION UNDER THE ACTION OF A VARIABLE FORCE—MOTION IN A RESISTING MEDIUM.

165 Definitions.................. 289
166 Newton's Laws of Motion.................................. 289
167. Remarks on Law I.. 290
168. Remarks on Law II. 291
169. Remarks on Law III...................................... 294
170. Two Laws of Motion in the French Treatises.............. 295
171. Motion of Particle under an Attractive Force.............. 295
172. Motion under the action of a Variable Repulsive Force..... 298
173. Motion under the action of an Attractive Force............ 299
174. Velocity acquired in Falling through a Great Height....... 300
175. Motion in a Resisting Medium............................ 302
176. Motion in the Air against the Action of Gravity........... 304
177. Motion of a Projectile in a Resisting Medium.............. 307
178. Motion against the Resistance of the Atmosphere.......... 308
179. Motion in the Atmosphere under a small Angle of Elevation 312
Examples....... 313

CHAPTER II.

CENTRAL FORCES.

ART. PAGE

180. Definitions. 321
181. A Particle under the Action of a Central Attraction. 321
182. The Sectorial Area Swept over by the Radius Vector. 325
183. Velocity of Particle at any Point of its Orbit. 325
184. Orbit when Attraction as the Inverse Square of Distance. . . 329
185. Suppose the Orbit to be an Ellipse. 333
186. Kepler's Laws. 335
187. Nature of the Force which acts upon the Planetary System. 335
Examples 338

CHAPTER III.

CONSTRAINED MOTION

188. Definitions. 345
189. Kinetic Energy or Vis Viva—Work. 345
190. To Find the Reaction of the Constraining Curve. 348
191. Point where Particle will leave Constraining Curve. 349
192. Constrained Motion Under Action of Gravity. 350
193. Motion on a Circular Arc in a Vertical Plane. 350
194. The Simple Pendulum. 352
195. Relation of Time, Length, and Force of Gravity 353
196. Height of Mountain Determined with Pendulum. 354
197. Depth of Mine Determined with Pendulum. 355
198. Centripetal and Centrifugal Forces. 356
199. The Centrifugal Force at the Equator. 358
200. Centrifugal Force at Different Latitudes. 359
201. The Conical Pendulum—The Governor. 361
Examples. 362

CHAPTER IV.

IMPACT.

202. An Impulsive Force. 370
203. Impact or Collision. 371
204. Direct and Central Impact. 372
205. Elasticity of Bodies—Coefficient of Restitution. 373

ART. PAGE
206. Direct Impact of Inelastic Bodies........................ 374
207. Direct Impact of Elastic Bodies........................... 375
208. Loss of Kinetic Energy in Impact of Bodies................ 378
209. Oblique Impact of Bodies.................................. 380
210. Oblique Impact of Two Smooth Spheres...................... 382
Examples... 383

CHAPTER V.

WORK AND ENERGY.

211. Definition and Measure of Work............................ 389
212. General Case of Work done by a Force...................... 390
213. Work on an Inclined Plane................................. 391
Examples... 393
214 Horse Power.. 395
215. Work of Raising a System of Weights....................... 396
Examples... 397
216. Modulus of a Machine...................................... 400
Examples... 401
217. Kinetic and Potential Energy—Stored Work.................. 404
Examples... 406
218. Kinetic Energy of a Rigid Body Revolving round an Axis.... 408
219. Force of a Blow... 411
220. Work of a Water Fall...................................... 412
221. The Duty of an Engine..................................... 414
222. Work of a Variable Force.................................. 415
223. Simpson's Rule.. 415
Examples... 417

CHAPTER VI.

MOMENT OF INERTIA.

224. Moment of Inertia... 429
225. Moments of Inertia relative to Parallel Axes or Planes.... 432
226. Radius of Gyration.. 434
227. Polar Moment of Inertia................................... 436
228. Moment of Inertia of a Solid of Revolution................ 437
229. Moment of Inertia about Axis Perpendicular to Geometric Axis.... 438
230. Moment of Inertia of Various Solid Bodies................. 440
231. Moment of Inertia of a Lamina with respect to any Axis.... 44

ART. PAGE
232. Principal Axes of a Body.............. 442
233. Products of Inertia..................................... 446
Examples.. 447

CHAPTER VII.

ROTATORY MOTION.

234. Impressed and Effective Forces.............................. 451
235. D'Alembert's Principle 452
236. Rotation of a Rigid Body about a Fixed Axis............ .. 454
237. The Compound Pendulum.......... 457
238. Length of Second's Pendulum Determined Experimentally.. 462
239. Motion of a Body when Unconstrained.................... 464
240. Centre of Percussion—Axis of Spontaneous Rotation........ 464
241. Principal Radius of Gyration Determined Practically....... 467
242. The Ballistic Pendulum................ 468
243. Motion of a Body about a Horizontal Axle through its Centre 470
244. Motion of a Wheel and Axle............................. 471
245. Motion of a Rigid Body about a Vertical Axis.............. 472
246. Body Rolling down an Inclined Plane...................... 473
247. Falling Body under an Impulse not through its Centre..... 475
Examples... 477

CHAPTER VIII.

MOTION OF A SYSTEM OF RIGID BODIES IN SPACE.

248. Equations of Motion obtained by D'Alembert's Principle ... 481
249. Independence of the Motions of Translation and Rotation... 482
250. Principle of the Conservation of the Centre of Gravity...... 485
251. Principle of the Conservation of Areas.................... 486
252. Conservation of Vis Viva or Energy......... 488
253. Composition of Rotations................................ 493
254. Motion of a Rigid Body referred to Fixed Axes. 494
255. Axis of Instantaneous Rotation........ 495
256. Angular Velocity about Axis of Instantaneous Rotation..... 496
257. Euler's Equations.................... 497
258. Motion about a Principal Axis through Centre of Gravity... 499
259. Velocity about a Principal Axis when Accelerating Forces $= 0$ 501
260. The Integral of Euler's Equations. 502
Examples....... 505

ANALYTIC MECHANICS.

PART I.

CHAPTER I.

FIRST PRINCIPLES.

1. Definitions.—*Analytic Mechanics* or ***Dynamics*** is the science which treats of the equilibrium and motion of bodies under the action of force. It is accordingly divided into two parts, *Statics* and *Kinetics*.

Statics treats of the equilibrium of bodies, and the conditions governing the forces which produce it.

Kinetics treats of the motion of bodies, and the laws of the forces which produce it.

The consideration that the properties of motion, velocity, and displacement may be treated apart from the particular forces producing them and independently of the bodies subject to them, has given rise to an auxiliary branch of Dynamics called *Kinematics*.*

Although Kinematics may not be regarded as properly included under Dynamics, yet this branch of science is so important and useful, and its application to Dynamics so immediate, that a portion of this work is devoted to its treatment.

---

* This name was given by Ampère.

Kinematics is the science of pure motion, without reference to matter or force. It treats of the properties of motion without regard to what is moving or how it is moved. It is an extension of pure geometry by introducing the idea of time, and the consequent idea of velocity.

2. Matter.—*Matter* is that which can be perceived by the senses, and which can transmit, and be acted upon by force. It has extension, resistance, and impenetrability.

A definition of matter which would satisfy the metaphysician is not required for this work. It is sufficient for us to conceive of it as capable of receiving and transmitting force; because it is in this aspect only that it is of importance in the present treatise.

3. Inertia.—By *Inertia* is meant that property of matter by which it remains in its state of rest or uniform motion in a right line unless acted upon by force. Inertia expresses the fact that a body cannot of itself change its condition of rest or motion. It follows that if a body change its state from rest to motion or from motion to rest, or if it change its direction from the natural rectilinear path, it must have been influenced by some external cause.

4. Body, Space, and Time.—*A Body* is a portion of matter limited in every direction, and is consequently of a determinate form and volume.

A Rigid Body is one in which the relative positions of its particles remain unchanged by the action of forces.

A Particle is a body indefinitely small in every direction, and though retaining its material properties may be treated as a geometric point.

Space is indefinite extension. *Time* is any limited portion of duration.

5. Rest and Motion.—A body is at *rest* when it constantly occupies the same place in space. A body is in *motion* when the body or its parts occupy successively different positions in space. But we cannot judge of the state of rest or motion of a body without referring it to the positions of other bodies; and hence rest and motion must be considered as necessarily *relative.*

If there were anything which we knew to be absolutely fixed in space, we might perceive absolute motion by change of place with reference to that object. But as we know of no such thing as absolute rest, it follows that all motion, as measured by us, must be relative; *i. e.*, must relate to something which we assume to be fixed. Hence the same thing may often be said to be at rest and in motion at the same time; for it may be at rest in regard to one thing, and in motion in regard to another. For example, the objects on a vessel may be at rest with reference to each other and to the vessel, while they are in motion with reference to the neighboring shore. So a man, punting his barge up the river, by leaning against a pole which rests on the bottom, and walking on the deck, is in motion relative to the barge, and in motion, but in a different manner, relative to the current, while he is at rest relative to the earth.

Motion is uniform when the body passes over equal spaces in equal times; otherwise it is *variable.*

6. Velocity.—*The velocity of a body is its rate of motion.* When the velocity is *constant*, it is measured by the space passed over in a unit of time. When it is *variable*, it is measured, at any instant, by the space over which the body would pass in a unit of time, were it to move, during that unit, with the same velocity that it has at the instant considered.

The speed of a railway train is, in general, variable. If we were to say, for example, that it was running at the rate of 30 miles an hour, we would not mean that it ran 30 miles during the last hour, nor that it would run 30 miles during the next hour. We would mean that, if it were to run for an hour with the speed which it now has, at the instant considered, it would pass over exactly 30 miles.

In order to have a uniform unit of velocity, it is customary to express it in *feet* and *seconds;* and when velocities are expressed in any other terms, they should be reduced to their equivalent value in feet and seconds. The unit velocity, therefore, is the velocity with which a body describes *one foot* in *one second*; other units may be taken where convenience demands, as miles and hours, etc.

When we speak of the *space* passed over by a body, we mean the *path* or *line* which a point in the body or which a particle describes.

7. Formulæ for Velocity.—If s be the space passed over by a particle in t units of time, and v the velocity, it is plain that, for *uniform velocity*, we shall have

$$v = \frac{s}{t}; \tag{1}$$

that is, we divide the whole space passed over by the time of the motion over that space.

If the velocity continually changes, equal increments are not described in equal times, and the velocity becomes a function of the time. But however much the velocity changes, it may be regarded as constant during the infinitesimal of time dt, in which time the body will describe the infinitesimal of space ds. Hence, denoting the velocity at any instant by v, we have

$$v = \frac{ds}{dt}. \tag{2}$$

In this case the velocity is the ratio of two infinitesimals. These two expressions for the velocity are true whether the particle be moving in a right, or in a curved, line.

8. Acceleration *is the rate of change of velocity.* It is a velocity increment. If the velocity is increasing, the acceleration is considered positive; if decreasing, it is negative.

Acceleration is said to be *uniform* when the velocity receives equal increments in equal times. Otherwise it is *variable.*

9. Measure of Acceleration.—Uniform acceleration is measured by the actual increase of velocity in a unit of time. Variable acceleration is measured, at any instant, by the velocity which would be generated in a unit of time, were the velocity to increase, during that unit, at the same rate as at the instant considered.

Calling f the acceleration, v the velocity, and t the time, we have, when the acceleration is uniform,

$$f = \frac{v}{t}. \tag{1}$$

However variable the acceleration is, it may be regarded as constant during the infinitesimal of time dt, in which time the increment of velocity will be dv. Hence, denoting the acceleration at the time t by f, we have

$$f = \frac{dv}{dt}. \tag{2}$$

We also have (Art. 8)

$$v = \frac{ds}{dt}$$

which in (2) gives

$$f = \frac{dv}{dt} = \frac{d}{dt} \cdot \frac{ds}{dt} = \frac{d^2s}{dt^2}. \tag{3}$$

That is, when the acceleration is variable it is measured, at any instant, by the derivative of the velocity regarded as a function of the time, or by the second derivative of the space regarded as a function of the time.

From (3) we get, by integration, when f is constant,

$$ft = \frac{ds}{dt} = v; \tag{4}$$

$$\tfrac{1}{2}ft^2 = s; \tag{5}$$

and

$$2fs = v^2, \tag{6}$$

which determine the velocity and space.

10. Geometric Representation of Velocity and Acceleration.—The velocity of a body may be conveniently represented geometrically in magnitude and direction by means of a straight line. Let the line be drawn from the point at which the motion is considered, and in the direction of motion at that point. With a convenient scale, let a length of the line be cut off that shall contain as many units of length as there are units in the velocity to be represented. The *direction* of this line will represent the direction of the motion, and its *length* will represent the velocity.

Also an *acceleration* may be represented geometrically by a straight line drawn in the direction of the velocity generated, and containing as many units of length as there are units of acceleration in the acceleration considered. Also, since an acceleration* is measured by the actual increase of velocity in the unit of time, the straight line which represents an acceleration in magnitude and direction will also completely represent the velocity generated in the unit of time to which the acceleration corresponds.

11. The Mass of a body or particle is the *quantity of matter* which it contains; and is proportional to the *Volume* and *Density* jointly. The *Density* may therefore be defined as the quantity of matter in a unit of volume.

Let M be the mass, ρ the density, and V the volume, of a homogeneous body. Then we have

$$M = V\rho, \tag{1}$$

if we so take our units that the unit of mass is the mass of the unit volume of a body of unit density.

---

* Uniform acceleration is here meant.

If the density varies from point to point of the body, we have, by the above formula, and the notation of the Integral Calculus,

$$M = \int \rho dV = \int\int\int \rho dx\, dy\, dz, \tag{2}$$

where ρ is supposed to be a known function of x, y, z.

In England the unit of mass is the imperial standard pound avoirdupois, which is the mass of a certain piece of platinum preserved at the standard office in London. On the continent of Europe the unit of mass is the gramme. This is known as the *absolute* or *kinetic* unit of mass.

12. The Quantity of Motion,* or the **Momentum** of a body moving without rotation is the product of its mass and velocity. A double mass, or a double velocity, would correspond to a double quantity of motion, and so on.

Hence, if we take as the unit of momentum the momentum of the unit of mass moving with the unit of velocity, the momentum of a mass M moving with velocity v is Mv.

13. Change of Quantity of Motion, or **Change of Momentum**, is proportional to the mass moving and the change of its velocity jointly. If then the mass remains constant the *change of momentum* is measured by the product of the mass into the change of velocity; and the *rate of change of momentum*, or *acceleration of momentum*, is measured by the product of the mass moving and the rate of change of velocity, that is, by the product of the mass moving and the acceleration (Art. 8). Thus, calling M the mass, we have for the *measure of the rate of change of momentum*,

$$M\frac{d^2s}{dt^2}.$$

---

* This phrase was used by Newton in place of the more modern term "Momentum."

14. Force.—*Force is any cause which changes, or tends to change, a body's state of rest or motion.*

A *force* always tends to produce motion, but may be prevented from actually producing it by the counteraction of an equal and opposite force. Several forces may so act on a body as to neutralize each other. When a body remains at rest, though acted on by forces, it is said to be in equilibrium; or, in other words, the forces are said to produce equilibrium.

What *force* is, in its nature, we do not know. Forces are known to us only by their *effects*. In order to measure them we must compare the effects which they produce under the same circumstances.

15. Static Measure of Force.—*The effect* of a force depends on: 1st, its *magnitude*, or *intensity;* 2d, its *direction; i. e.*, the direction in which it tends to move the body on which it acts; and 3d, its *point of application; i. e.*, the point at which the force is applied.

The effect of a force is *pressure*, and may be expressed by the weight which will counteract it. Every force, statically considered, is a pressure, and hence has magnitude, and may be measured. A force may produce motion or not, according as the body on which it acts is or is not free to move. For example, take the case of a body resting on a table. The same force which produces pressure on the table would cause the body to fall toward the earth if the table were removed.

The cause of this pressure or motion is gravity, or the force of attraction in the earth. In the first case the attraction of the earth produces a pressure; in the second case it produces motion. Now either of these, viz., the pressure which the body exerts when at rest, or the quantity of motion it acquires in a unit of time, may be taken as a means of measuring the magnitude of the force of attraction that the earth exerts on the body. The former is called the *static* method, and the forces are called *static forces;* the latter is called the *kinetic* method, and the forces are called *kinetic forces.* *Weight* is the name given to the pressure which the attraction of the earth causes a body to exert. Hence, since static forces produce pressure, we may take, as the *unit of force, a pressure of one pound* (Art. 11).

Therefore, *the magnitude of a force may be measured statically by the pressure it will produce upon some body, and expressed in pounds.* This is called the *Static measure of force*, and its unit, one pound, is called the *Gravitation unit of force.*

16. Action and Reaction are always equal and opposite.—This is a law of nature, and our knowledge of it comes from experience. If a force act on a body held by a fixed obstacle, the latter will oppose an equal and contrary resistance. If the force act on a body free to move, motion will ensue; and, in the act of moving, the *inertia* of the body will oppose an equal and contrary resistance. If we press a stone with the hand, the stone presses the hand in return. If we strike it, we receive a blow by the act of giving one. If we urge it so as to give it motion, we lose some of the motion which we should give to our limbs by the same effort, if the stone did not impede them. In each of these cases there is a reaction of the same kind as the action, and equal to it.

17. Method of Comparing Forces.—Two forces are equal when being applied in opposite directions to a particle they maintain equilibrium. If we take two *equal* forces, and apply them to a particle in the *same* direction, we obtain a force double of either; if we unite *three* equal forces we obtain a *triple* force; and so on. So that, in general, to compare or measure forces, we have only to adopt the same method as when we compare or measure any quantities of the same kind; that is, we must take some known force as the *unit of force*, and then express, in numbers, the relation which the other forces bear to this measuring unit. For example, if one pound be the unit of force (Art. 15), a force of 12 pounds is expressed by 12; and so on.

18. Representation of Forces by Symbols and Lines.—If P. Q. R., etc., represent forces, they are numbers expressing the number of times which the concrete unit of force is contained in the given forces.

Forces may be represented geometrically by right lines; and this mode of representation has the advantage of giving the direction, magnitude, and point of application of each force. Thus, draw a line in the direction of the given force; then, having selected a unit of length, such as an inch, a foot, etc., measure on this line as many units of length as the given force contains units of weight. The *magnitude* of the force is represented by the measured length of the line; its *direction* by the direction in which the line is drawn; and its *point of application* by the point from which the line is drawn.*

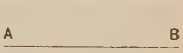

Fig. 1.

Thus, let the force P act at the point A, in the direction AB, and let AB represent as many units of length as P contains units of force; then the force P is represented geometrically by the line AB; for the force acts in the direction from A to B; its point of application is at A, and its magnitude is represented by the length of the line AB.

19. Measure of Accelerating Forces.—From our definition of *force* (Art. 14), it is clear that, when a single force acts upon a particle, perfectly free to move, it must produce motion; and hence the force may be represented to us by the motion it has produced. But motion is measured in terms of velocity (Art. 6), and consequently the velocity communicated to, or impressed upon, a particle, in a given time, may be taken as a measure of the force. That is, if the same particle moves along a right line so that its velocity is increased at a constant rate, it will be acted upon by a constant force. If a certain constant force, acting for a second on a given particle, generate a velocity of 32.2 feet per second, a double force, acting for one second on the same particle, would generate a velocity of 64.4 feet per second; a triple force would generate a velocity of 96.6 feet per second, and so on.

---

* Forces, velocities, and accelerations are *directed quantities*, and so may be represented by a line, in direction and magnitude, and may be compounded in the same way as *vectors*.

If anything has magnitude and direction, the magnitude and direction taken together constitute a *vector*.

If the rate of increase of the velocity, (*i. e.*, the acceleration), of the particle is not uniform, the force acting on it is not uniform, and the magnitude of the force, at any point of the particle's path, is measured by the acceleration of the particle at this point. Hence, since one and the same particle is capable of moving with all possible accelerations, all *forces may be measured by the velocities they generate in the same or equal particles in the same or equal times.* When forces are so measured they are called *Accelerating Forces.*

20. Kinetic or Absolute Measure of Force.*—Let n equal particles be placed side by side, and let each of them be acted on uniformly for the same time, by the same force. Each particle, at the end of this time, will have the same velocity. Now if these n separate particles are all united so as to form a body of n times the mass of each particle, and if each one of them is still acted on by the same force as before, this body, at the end of the time considered, will have the same velocity that each separate particle had, and will be acted on by n times the force which generated this velocity in the particle. Comparing a single particle, then, with the body whose mass is n times the mass of this particle, we see that, to produce the same velocity in two bodies by forces acting on them for the same time, the magnitudes of the forces must be proportional to the masses on which they act.* Hence, generally, since force varies as the velocity when the mass is constant (Art. 19). and varies as the mass when the velocity is constant, we have, by the ordinary law of proportion, when both are changed, force varies as the product of the mass acted upon and the velocity generated in a given time ; that is, it varies as the quantity of motion (Art. 13) it produces in a given mass in a given time. If the force be variable, the rate of change of velocity is variable (Art. 19), and hence the force varies as the product of the mass on which it acts and the *rate of change* of velocity, *i. e.*, it varies as the *acceleration of the momentum* (Art. 13). Therefore, if any force P act on a mass M, we have

---

* Arts. 20, 21, 22, and 25, treat of the Kinetic measure of force, and may be omitted till Part III is reached; but it is convenient to present them once for all, and, for the sake of reference and comparison, to place them with the Static measure of force at the beginning of the work.

$$P \propto Mf; \qquad (1)$$

or, in the form of an equation

$$P = kMf, \qquad (2)$$

where k is some constant.

If the unit of force be taken as that force which, acting on the unit of mass for the unit of time, generates the unit of velocity, then if we put M equal to unity, *i. e.*, take the unit of mass, and f equal to unity, *i. e.*, take the unit of acceleration, we must have the force producing the acceleration equal to the unit of force, or P equal to unity.

---

* Minchin's Statics, p. 5.

Hence k must also be equal to unity, and we have the equation,

$$P = Mf. \tag{3}$$

Therefore, the *Kinetic or Absolute measure of a force is the rate of change or acceleration* of momentum it produces in a unit of time.*

If the force is constant, (3) becomes by (1) of Art. 9,

$$P = \frac{Mv}{t}. \tag{4}$$

And if the force is variable, (3) becomes by (3) of Art. 9,

$$P = M\frac{d^2s}{dt^2}. \tag{5}$$

21. The Absolute or Kinetic Unit of Force.—A second, a foot, and a pound being the units of time, space, and mass, respectively (Arts. 6 and 11), we are required to find the corresponding unit of force that the above equation may be true. *The unit of force is that force which, acting for one second, on the mass of one pound, generates in it a velocity of one foot per second.* Now, from the results of numerous experiments, it has been ascertained that if a body, weighing one pound, fall freely for one second at the sea level, it will acquire a velocity of about 32.2 feet per second; *i. e.*, a force equal to the weight of a pound, if acting on the mass of a pound, at the sea level, generates in it in one second, if free to move, a velocity of nearly 32.2 feet per second. It follows, therefore, that a force of $\frac{1}{32.2}$ of the weight of a pound, if acting on the mass of a pound, at the sea level, generates in it in one second, if free to move, a velocity of one foot per second; and hence the unit of force is $\frac{1}{32.2}$ of the weight of a pound, or rather less than the weight of half an ounce avoirdupois; so that half an ounce, acting on the mass of a pound for one second, will give to it a velocity of one foot per second. This is the *British absolute kinetic** unit of force.

---

* See Tait and Steele's Dynamics of a Particle, p. 43.

In order that Eq. 3 (Art. 20) may be universally true when a second, a foot, and a pound are the units of time, space, and mass respectively, all forces must be expressed in terms of this unit.

22. Three Ways of Measuring Force.—(1.) If a force does not produce motion it is measured by the pressure it produces, or the number of pounds it will support (Art. 15). This is the measure of *Static Force, and its unit is the weight of a pound.*

(2.) If we consider forces as always acting on a *unit of mass*, and suppose that there are no forces acting in the opposite direction, then these forces will be measured simply by the *velocities* or *accelerations* which they generate in a given time. This is the measure of *Accelerating Force, and its unit is that force which, acting on the unit of mass, during the unit of time, generate the unit of velocity;* hence (Art. 21), *the unit of force is the force which, acting on one pound of mass for one second, generates a velocity of one foot per second.*

(3.) If forces act on *different masses*, and produce motion in them, and we consider as before that there are no forces acting in the opposite direction, then the forces are *measured by the quantity of motion, or by the acceleration of momentum generated in a unit of time* (Art. 20). This is the measure of *Moving Force*, and its *unit* (Art. 21) *is the force which, acting on one pound of mass for one second, generates a velocity of one foot per second.*

---

* Introduced by Gauss.

It must be understood that when we speak of static, accelerating, or moving forces, we do not refer to different kinds of force, but only to force as measured in different ways.

23. Meaning of g in Dynamics.—The most important case of a constant, or very nearly constant, force is gravity at the surface of the earth. The force of gravity is so nearly constant for places near the earth's surface, that falling bodies may be taken as examples of motion under a constant force. A stone, let fall from rest, moves at first very slowly. During the first tenth of a second the velocity is very small. In one second the stone has acquired a velocity of *about* 32 feet per second.

A great number of experiments have been made to ascertain the exact velocity which a body would acquire in one second under the action of gravity, and freed from the resistance of the air. The most accurate method is indirect, by means of the pendulum. The result of pendulum experiments made at Leith Fort, by Captain Kater, is, that the velocity acquired by a body falling unresisted for one second is, at that place, 32.207 feet per second. The velocity acquired in one second, or the acceleration (Art. 8), of a body falling freely in vacuo, is found to vary slightly with the latitude, and also with the elevation above the sea level. In London it is 32.1889 feet per second. In latitude 45°, near Bordeaux, it is 32.1703 feet per second.

This acceleration is usually denoted by g; and when we say that at any place g is equal to 32, we mean that the velocity generated per second in a body falling freely* under the action of gravity at that place, is a velocity of 32 feet per second. The average value of g for the whole of Great Britain differs but little from 32.2; and hence the numerical value of g for that country is taken to be 32.2.

---

* A body is said to be moving *freely* when it is acted upon by no forces except those under consideration.

The formula, deduced from observation, and a certain theory regarding the figure and density of the earth, which may be employed to calculate the most probable value of the apparent force of gravity, is

$$g = G(1 + .005133 \sin^2 \lambda),$$

where G is the apparent force of gravity on a unit mass at the equator, and g the force of gravity in any latitude λ; the value of G, in terms of the British absolute unit, being 32.088. (See Thomson and Tait, p. 226.)

24. Gravitation Units of Force and Mass.—If in (3) of Art. 20, we put for P, the weight W of the body, and write g for f since we know the acceleration is g, (3) becomes

$$W = mg. \qquad (1)$$

$$\therefore \quad m = \frac{W}{g}. \qquad (2)$$

and hence $\frac{W}{g}$ may be taken as the measure of the mass.

In gravitation measure forces are measured by the pressure they will produce, and the unit of force is one pound (Art. 15), *and the unit of mass is the quantity of matter in a body which weighs g pounds at that place where the acceleration of gravity is g.*

This definition gives a unit of mass which is constant at the same place, but changes with the locality; *i. e.*, its *weight* changes with the locality while the *quantity of matter* in it remains the same. Thus, the unit of mass would weigh at Bordeaux 32.1703 pounds (Art. 23), while at Leith Fort it would weigh 32.207 pounds. Let m be the mass of a body which weighs w pounds. The quantity of matter in this body remains the same when carried from place to place. If it were possible to transport it to another planet its *mass* would not be altered, but its *weight* would be very different. Its weight wherever placed would vary directly as the force of gravity; but the acceleration also would vary directly as the force of gravity. If placed on the sun, for example, it would weigh about 28 times as much as on the surface of the earth; but the acceleration on the sun would also be 28 times as much as on the surface of the earth; that is, the *ratio* of the weight to the acceleration, anywhere in the universe is constant, and hence $\frac{W}{g}$, which is the numerical value of m (Eq. 2), is constant for the same mass at all places.

25. Comparison of Gravitation and Absolute Measure. The pound weight has been long used for the measurement of *force* instead of *mass*, and is the recognized standard of reference. It came into general use because it afforded the most ready and simple method of estimating forces. The pressure of steam in a boiler is always reckoned in pounds per square inch. The tension of a string is estimated in pounds: the force necessary to draw a train of cars, or the pressure of water against a lock-gate, is expressed in pounds. Such expressions as "a force of 10 pounds," or "a pressure of steam equal to 50 pounds on the inch," are of every day occurrence. Therefore this method of measuring forces is eminently convenient in practice. For this reason, and because it is the one used by most engineers and writers of mechanics, we shall adopt it in this work, and adhere to the measurement of force by pounds, and give all our results in the usual gravitation measure. In this measure it is convenient to represent the mass of a body weighing W pounds by the fraction $\frac{W}{g}$ (Art. 24), so that (3) of Art. 20 becomes

$$P = \frac{W}{g}f. \qquad (1)$$

To do so it will only be necessary to assume that the unit of mass is the quantity of matter in a body weighing g pounds, and changes in weight in the same proportion that g changes (Art. 24).

Of course, the units of mass and force in (3) of Art. 20 may be either absolute or gravitation units. If *absolute*, the unit of mass is one pound (Art. 11), and the unit of force is $\frac{1}{g}$ pounds (Art. 21). If *gravitation*, the units are g times as great; *i. e.*, the unit of mass is g pounds (Art. 24), and the unit of force is one pound (Art. 15).

The advantage of the gravitation measure is, it enables us to express the force in *pounds*, and furnishes us with a constant numerical representative for the same quantity of matter; that is to say, a mass represented by 20 on the equator would be represented by 20, at the pole or on the sun. Hence, in (1), P is the *static measure* of any moving force [Art. 22, (3)], W is the *weight* of the body in pounds, g the *acceleration of gravity* (Art. 23), $\frac{W}{g}$ the *mass* upon which the force acts [(2) of Art. 24], and which is free to move under the action of P, the unit of mass being the mass weighing g pounds, and f the *acceleration* which the force P produces in the mass.

EXAMPLES

1. Compare the velocities of two points which move uniformly, one through 5 feet in half a second, and the other through 100 yards in a minute. *Ans.* As 2 is to 1.

2. Compare the velocities of two points which move uniformly, one through 720 feet in one minute, and the other through $3\frac{1}{2}$ yards in three-quarters of a second.

Ans. As 6 is to 7.

3. A railway train travels 100 miles in 2 hours; find the average velocity in feet per second. *Ans.* $73\frac{1}{3}$.

4. One point moves uniformly round the circumference of a circle, while another point moves uniformly along the diameter; compare their velocities.

Ans. As π is to 1.

5. Supposing the earth to be a sphere 25000 miles in circumference, and turning round once in a day, determine the velocity of a point at the equator.

Ans. $1527\frac{7}{9}$ ft. per sec.

6. A body has described 50 feet from rest in 2 seconds, with uniform acceleration; find the velocity acquired.

From (5) of Art. 9 we have

$$f = 25;$$

and from (4) we have $ft = v$;

$$\therefore \quad v = 50.$$

7. Find the time it will take the body in the last example to move over the next 150 feet.

From (5) of Art. 9 we have

$$s = \tfrac{1}{2}ft^2; \quad \therefore \quad \text{etc.}$$

Ans. 2 seconds.

8. A body, moving with uniform acceleration, describes 63 feet in the fourth second; find the acceleration.

Ans. 18.

9. A body, with uniform acceleration, describes 72 feet while its velocity increases from 16 to 20 feet per second; find the whole time of motion, and the acceleration.

Ans. 20 seconds; 1.

10. A body, in passing over 9 feet with uniform acceleration, has its velocity increased from 4 to 5 feet per second; find the whole space described from rest, and the acceleration. *Ans.* 25 feet; $\frac{1}{2}$.

11. A body, uniformly accelerated, is found to be moving at the end of 10 seconds with a velocity which, if continued uniformly, would carry it through 45 miles in the next hour; find the acceleration. *Ans.* $6\frac{3}{5}$.

12. Find the mass of a straight wire or rod, the density of which varies directly as the distance from one end.

Take the end of the rod as origin; let $a =$ its length; let the distance of any point of it from that end $= x$; and let $\omega =$ the area of its transverse section, and $k =$ the density at the unit's distance from the origin. Then

$$dV = \omega dx; \quad \text{and} \quad \rho = kx;$$

and (2) of Art. 11 becomes

$$M = \int_0^a k\omega x\, dx = \frac{k\omega a^2}{2}.$$

13. Find the mass of a circular plate of uniform thickness, the density of which varies as the distance from the centre.

Ans. $\frac{2}{3}\pi kha^3$, where a is the radius, k the density at the unit's distance, and h the thickness.

14. Find the mass of a sphere, whose density varies inversely as the distance from the centre.

Ans. $2\pi\rho a^3$, where ρ is the density of the outside stratum.

STATICS (REST).

CHAPTER II.

THE COMPOSITION AND RESOLUTION OF CONCURRING FORCES—CONDITIONS OF EQUILIBRIUM.

26. Problem of Statics.—The primary conception of *force* is that of a cause of motion (Art. 14). If only one force acts on a particle it is clear that the particle cannot remain at rest. In statics it is only the *tendency* which forces have to produce motion that is considered. There must be at least two forces in statics; and they are considered as acting so as to counteract each other's tendency to cause motion, thereby producing a state of equilibrium in the bodies to which they are applied. The forces which act upon a body may be in equilibrium, and yet motion exist; but in such cases the motion is uniform. Hence there are two kinds of equilibrium, the one relating to bodies at rest, the other relating to bodies in motion. The former is sometimes called *Static* Equilibrium and the latter *Kinetic* (or Dynamic*) Equilibrium. *The problem of statics is to determine the conditions under which forces act when they keep bodies at rest.*

27. Concurring and Conspiring Forces.—Resultant.—When several forces have a common point of application they are called *concurring* forces; when they act at the same point and along the same right line they are called *conspiring* forces.

The *resultant* of two or more forces is that force which singly will produce the same effect as the forces themselves when acting together. The individual forces, when considered with reference to this resultant, are called

---

* Gregory's Mechanics, p. 14.

components. The process of finding the resultant of several forces is called *the composition of forces.*

28. Composition of Conspiring Forces.—Condition of Equilibrium.—When two or more conspiring forces act in the same direction, it is evident that the resultant force is equal to their sum, and acts in the same direction.

When two conspiring forces act in opposite directions their resultant force is equal to their difference, and acts in the direction of the greater component.

When several conspiring forces act in different directions the resultant of the forces acting in one direction equals the sum of these forces, and acts in the same direction; and so of the forces acting in the opposite direction. Therefore, the resultant of all the forces is equal to the difference of these sums, and acts in the direction of the greater sum. Hence, if the forces acting in one direction are reckoned positive, and those in the opposite direction negative, their resultant is equal to their algebraic sum; its sign determining the direction in which it acts. Thus, if P_1, P_2, P_3, etc., are the conspiring forces, some of which may be positive and the others negative, and R is the resultant, we have

$$R = P_1 + P_2 + P_3 + \text{etc.} = \Sigma P, \qquad (1)$$

in which Σ denotes the algebraic sum of the terms similar to that written immediately after it.

Cor.—The condition that the forces may be in equilibrium is that their resultant, and therefore their algebraic sum, must vanish. Hence, when the forces are in equilibrium we must have $R = 0$; therefore (1) becomes

$$P_1 + P_2 + P_3 + \text{etc.} = \Sigma P = 0. \qquad (2)$$

29. Composition of Velocities.—*If a particle be moving with two uniform velocities represented in magnitude and direction by the two adjacent sides of a parallelogram, the resultant velocity will be represented in magnitude and direction by the diagonal of the parallelogram.*

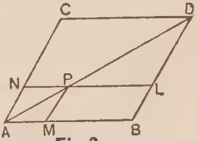

Fig.2

Let the particle move with a uniform velocity v, which acting alone will take it in one second from A to B, and with a uniform velocity v', which acting alone will take it in one second from A to C; at the end of one second the particle will be found at D, and AD will represent in magnitude and direction the resultant of the velocities represented by AB and AC.

Suppose the particle to move uniformly along a straight tube which starts from AB, and moves uniformly parallel to itself with its extremity in AC. When the particle starts from A the tube is in the position AB. When the particle has moved over any part of AB, the end of the tube has moved over the same part of AC, and the particle is on the line AD. For example, let AM be the $\frac{1}{n}$th part of AB, and AN be the $\frac{1}{n}$th part of AC; while the particle moves from A to M, the end A with the tube AB will move from A to N, and the particle will be at P, the tube occupying the position NL, and PM being parallel and equal to AN. P can be proved to be on the diagonal AD as follows:

$$\text{AM} : \text{MP} :: \frac{\text{AB}}{n} : \frac{\text{AC}}{n} :: \text{AB} : \text{AC} (= \text{BD});$$

therefore P lies on the diagonal AD. Also since

$$\text{AM} : \text{AB} :: \text{AP} : \text{AD},$$

the resultant velocity is uniform. Hence, the diagonal AD represents in magnitude and direction the resultant of the velocities represented by AB and AC.

This proposition is known as *the Parallelogram of Velocities.*

30. Composition of Forces.—From the *Parallelogram of Velocities* the *Parallelogram of Forces* follows immediately. Since two simultaneous velocities, AB and AC, of a particle, result in a single velocity, AD, and since these three velocities may be regarded as the measures of three separate forces all acting for the same time (Art. 19), it follows that the effect produced on a particle by the combined action, for the same time, of two forces may be produced by the action, for the same time, of a single force, which is therefore called the *resultant* of the other two forces; and these forces are represented in magnitude and direction by AB, AC, and AD. (See Minchin, p. 7, also Garnett's Dynamics, p. 10.)

Hence if two concurring forces be represented in magnitude and direction by the adjacent sides of a parallelogram, their resultant will be represented in magnitude and direction by the diagonal of the parallelogram. Care must be taken in constructing the parallelogram of forces that the components both act *from* the angle of the parallelogram from which the diagonal is drawn.

This proposition has been proved in various ways. It was enunciated in its present form by Sir Isaac Newton, and by Varignon, the celebrated mathematician, in the year 1687, probably independent of each other. Since that time various proofs of it have been given by different mathematicians. One work gives a discussion, more or less complete, of 45 other proofs. A noted analytic proof is given by M. Poisson. (See Price's Cal., Vol. III, p. 19). Some authors object to proving the parallelogram of forces by means of the parallelogram of velocities. (See Gregory's Mechanics, p. 14.) The student who wants other proofs is referred to Duchayla's proof as found in Todhunter's Statics, p. 7, and in Galbraith's Mechanics, p. 7, and in many other works; or to Laplace's proof. (See Mécanique Celeste, Liv. I, chap. 1.)

If θ be the angle between the sides of the parallelogram, AB and AC (Fig. 2), and P and Q represent the two component forces acting at A, and R represent the resultant, AD, we have from trigonometry,

$$R^2 = P^2 + Q^2 + 2PQ \cos \theta \qquad (1)$$

an equation which gives the *magnitude* of the resultant of two forces in terms of the magnitudes of the two forces and the angle between their directions, the forces being represented by two lines, both drawn *from* the point at which they act.

Cor.—If $\theta = 90°$, and α and β be the angles which the direction of R makes with the directions of P and Q, we have from (1)

$$R^2 = P^2 + Q^2. \qquad (2)$$

Also

$$\left.\begin{aligned} \cos \alpha &= \frac{P}{R}, \\ \cos \beta &= \frac{Q}{R}; \end{aligned}\right\} \qquad (3)$$

from which the magnitude and direction of the resultant are determined.

31. Triangle of Forces.—*If three concurring forces be represented in magnitude and direction by the sides of a triangle, taken in order, they will be in equilibrium.*

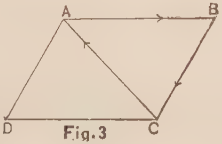

Fig. 3

Let ABC be the triangle whose sides, taken in order, represent in magnitude and direction three forces applied at the point A. Complete the parallelogram ABCD. Then the forces, AB and BC, applied at A, are expressed by AB and AD (since AD is equal and parallel to BC). But the resultant of AB and AD is AC, acting in the direction AC. Therefore the three forces represented by AB, BC, and CA, are equivalent to two forces, AC and CA, the former acting from A towards C and the latter from C towards A, which, being equal and opposite, will clearly balance each other. Therefore the three forces represented by AB, BC, and CA, acting at the point A, will be in equilibrium.

It should be observed that though BC represents the *magnitude* and *direction* of the component, it is not in the *line* of its action, because the three forces act at the point A.

The converse of this is also true ; viz., If three concurring forces are in equilibrium, they may be represented in magnitude and direction by the sides of a triangle, drawn parallel respectively to the directions of the forces.

Thus, if AB and BC represent two forces in magnitude and direction, AC will represent the resultant, and hence to produce equilibrium the resultant force AC must be opposed by an equal and opposite force CA. Therefore, the three forces in equilibrium will be represented by AB, BC, and CA.

Cor.—When three concurring forces are in equilibrium, each is equal and directly opposite to the resultant of the other two.

32. Relations between Three Concurring Forces in Equilibrium.—Since the sides of a plane triangle are as the sines of the opposite angles, we have (Fig. 3)

$$\begin{aligned} \mathrm{AB} : \mathrm{BC}\ (\text{or AD}) : \mathrm{AC}\ &:: \sin \mathrm{ACB} : \sin \mathrm{BAC} : \sin \mathrm{ABC} \\ &:: \sin \mathrm{DAC} : \sin \mathrm{BAC} : \sin \mathrm{BAD}. \end{aligned}$$

Hence, calling P, Q, and R, the forces represented by AB, AD, and AC, and denoting the angles between the directions of the forces P and Q, Q and R, and R and P, by $\widehat{PQ}$, $\widehat{QR}$, and $\widehat{RP}$, respectively, we have

$$\frac{P}{\sin \widehat{QR}} = \frac{Q}{\sin \widehat{RP}} = \frac{R}{\sin \widehat{PQ}}. \qquad (1)$$

Therefore, when three concurring forces are in equilibrium they are respectively in the same proportion as the sines of the angles included between the directions of the other two.

33. The Polygon of Forces.—*If any number of concurring forces be represented in magnitude and direction by the sides of a closed polygon taken in order, they will be in equilibrium.*

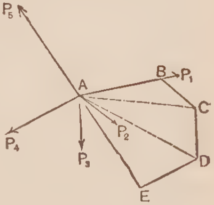

Fig.4

Let the forces be represented in magnitude and direction by the lines AP_1, AP_2, AP_3, AP_4, AP_5. Take AB to represent AP_1, through B draw BC equal and parallel to AP_2; the resultant of the forces AB and BC, or AP_1 and AP_2 is represented by AC (Art. 31). Of course the force, BC, acts at A and is parallel to BC. Again through C draw CD equal and parallel to AP_3, the resultant of AC and CD, or AP_1, AP_2, and AP_3 is AD. Also through D draw DE equal and parallel to AP_4, the resultant of AD and DE, or AP_1, AP_2, AP_3, and AP_4 is AE. Now if AE is equal and opposite to AP_5 the system is in equilibrium (Art. 18). Hence the forces represented by AB, BC, CD, DE, EA will be in equilibrium.

Cor. 1.—Any one side of the polygon represents in magnitude and direction the resultant of all the forces represented by the remaining sides.

Cor. 2.—If the lines representing the forces do not form a closed polygon the forces are not in equilibrium; in this case the last side, AE, taken from A to E, or that which is required to close up the polygon, represents in magnitude and direction the resultant of the system.

34. Parallelopiped of Forces.—*If three concurring forces, not in the same plane, are represented in magnitude and direction by the three edges of a parallelopiped, then the resultant will be represented in magnitude and direction by the diagonal; conversely, if the diagonal of a parallelopiped represents a force, it is equivalent to three forces represented by the edges of the parallelopiped.*

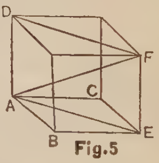

Fig.5

Let the three edges AB, AC, AD of the parallelopiped represent the three forces, applied at A. Then the resultant of the forces AB and AC is AE, the diagonal of the face ABCE; and the resultant of the forces AE and AD is AF, the diagonal of the parallelogram ADFE. Hence AF represents the resultant of the three forces AB, AC, and AD.

Conversely, the force, AF, is equivalent to the three components AB, AC, and AD.

Let P, Q, S represent the three forces AB, AC, AD; R, the resultant; α, β, γ, the angles which the direction of R makes with the directions of P, Q, S, and suppose the forces to act at right angles with each other. Then since

$$\overline{AF}^2 = \overline{AB}^2 + \overline{AC}^2 + \overline{AD}^2,$$

we have
$$R^2 = P^2 + Q^2 + S^2; \tag{1}$$

also,
$$\left.\begin{aligned} \cos\alpha &= \frac{P}{R}, \\ \cos\beta &= \frac{Q}{R}, \\ \cos\gamma &= \frac{S}{R}; \end{aligned}\right\} \tag{2}$$

from which the magnitude and direction of the resultant are determined.

EXAMPLES.

1. Three forces of 5 lbs., 3 lbs., and 2 lbs., respectively, act upon a point in the same direction, and two other forces of 8 lbs. and 9 lbs. act in the opposite direction. What single force will keep the point at rest? *Ans.* 7 lbs.

2. Two forces of $5\frac{1}{2}$ lbs. and $3\frac{1}{2}$ lbs., applied at a point, urge it in one direction; and a force of 2 lbs., applied at the same point, urges it in the opposite direction. What additional force is necessary to preserve equilibrium? *Ans.* 7 lbs.

3. If a force of 13 lbs. be represented by a line of $6\frac{1}{2}$ inches, what line will represent a force of $7\frac{1}{2}$ lbs.? *Ans.* $3\frac{3}{4}$ inches.

4. Two forces whose magnitudes are as 3 to 4, acting on a point at right angles to each other, produce a resultant of 20 lbs.; required the component forces. *Ans.* 12 lbs. and 16 lbs.

5. Let ABC be a triangle, and D the middle point of the side BC. If the three forces represented in magnitude and direction by AB, AC, and AD, act upon the point A; find the direction and magnitude of the resultant.

Ans. The direction is in the line AD, and the magnitude is represented by 3AD.

6. When $P = Q$ and $\theta = 60°$, find R. *Ans.* $R = P\sqrt{3}$.

7. When $P = Q$ and $\theta = 135°$, find R. *Ans.* $R = P\sqrt{2 - \sqrt{2}}$.

8. When $P = Q$ and $\theta = 120°$, find R. *Ans.* $R = P$.

9. If $P = Q$, show that their resultant $R = 2P \cos \frac{\theta}{2}$.

10. If $P = 8$, and $Q = 10$, and $\theta = 60°$, find R.
Ans. $R = 2\sqrt{61}$.

11. If $P = 144$, $R = 145$, and $\theta = 90°$, find Q.
Ans. $Q = 17$.

12. Two forces of 4 lbs. and $3\sqrt{2}$ lbs. act at an angle of 45°, and a third force of $\sqrt{42}$ lbs. acts at right angles to their plane at the same point; find their resultant.
Ans. 10 lbs.

35. Resolution of Forces.—*By the resolution of forces is meant the process of finding the components of given forces.* We have seen (Art. 30) that two concurring forces, P and Q = AB and AC, (Fig. 2) are equivalent to a single force R = AD; it is evident then that the single force, R, acting along AD, can be replaced by the two forces, P and Q, represented in magnitude and direction by two adjacent sides of a parallelogram, of which AD is the diagonal.

Since an infinite number of parallelograms, of each of which AD is the diagonal, can be constructed, it follows that a single force, R, can be resolved into two other forces in an infinite number of ways.

Also, each of the forces AB, AC, may be resolved into two others, in a way similar to that by which AD was resolved into two; and so on to any extent. Hence, a single force may be resolved into any number of forces, whose combined action is equivalent to the original force.

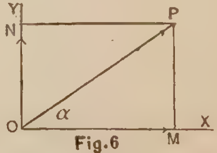

Fig. 6

Cor.—The most convenient components into which a force can be resolved are those whose directions are at right angles to each other. Thus, let OX and OY be any two lines at right angles to each other, and P any force acting at O in the plane XOY. Then completing the rectangle $OMPN$ we find the components of P along the axes OX and OY to be OM and ON, which denote by X and Y. Then we have clearly

$$\left.\begin{aligned} X &= P \cos \alpha, \\ Y &= P \sin \alpha; \end{aligned}\right\} \qquad (1)$$

where α is the angle which the direction of P makes with OX. These components X and Y are called the rectangular components. The rectangular component of a force, P, along a right line is $P \times$ cosine of angle between line and direction of P.

In strictness, when we speak of the component of a given force along a certain line, it is necessary to mention the other line along which the other component acts. In this work, unless otherwise expressed, the component of a force along any line will be understood to be its *rectangular component*; *i.e.*, the resolution will be made along this line and the line perpendicular to it.

36. To find the Magnitude and Direction of the Resultant of any number of Concurring Forces in one Plane.—When there are several concurring forces, the condition of their equilibrium may be expressed as in Art. 33, Cors. 1 and 2. But in practice we obtain much simpler results by using the principle of the *Resolution of Forces* (Art. 35), than those given by the principle of *Composition of Forces.*

Let O be the point at which all the forces act. Through O draw the rectangular axes XX', YY'. Let P_1, P_2, P_3, etc., be the forces and α_1, α_2, α_3, etc., be the angles which their directions make with the axis of x.

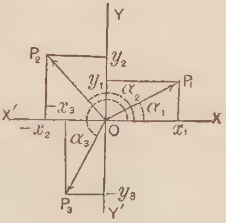

Fig. 7

Now resolve each force into its two components along the axes of x and y. Then the components along the axis of x (x-components) are (Art. 35, Cor.), $P_1 \cos \alpha_1$, $P_2 \cos \alpha_2$, $P_3 \cos \alpha_3$, etc., and those along the axis of y are $P_1 \sin \alpha_1$, $P_2 \sin \alpha_2$, $P_3 \sin \alpha_3$, etc.; and therefore if X and Y denote the algebraic sum of the x-components and y-components respectively, we have

$$\left.\begin{aligned} X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \text{etc.} \\ = \Sigma P \cos \alpha, \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} Y = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + \text{etc.} \\ = \Sigma P \sin \alpha. \end{aligned}\right\} \quad (2)$$

Let R be the resultant of all the forces acting at O, and θ the angle which it makes with the axis of x; then resolving R into its x- and y-components, we have

$$\left.\begin{aligned} R \cos \theta = X = \Sigma P \cos \alpha, \\ R \sin \theta = Y = \Sigma P \sin \alpha. \end{aligned}\right\} \quad (3)$$

$$\therefore \quad R^2 = X^2 + Y^2; \ \tan \theta = \frac{Y}{X}, \quad (4)$$

which determines the magnitude and direction of the resultant.

Sch.—Regarding OX and OY as positive and OX^1 and OY^1 as negative as in Anal. Geom., we see that Ox_1, Oy_1, Oy_2 are positive, and Ox_2, Ox_3, Oy_3 are negative. The *forces* may always be considered as positive, and hence the signs of the components in (1) and (2) will be the same as those of the trigonometric functions. Thus, since α_2 is $> 90°$ and $< 180°$ its sine is positive and cosine is negative; since α_3 is $> 180°$ and $< 270°$ both its sine and cosine are negative.

37. The Conditions of Equilibrium for any number of Concurring Forces in one Plane.—For the equilibrium of the forces we must have $R = 0$. Hence (4) of Art. 36 becomes

$$X^2 + Y^2 = 0. \quad (1)$$

Now (1) cannot be satisfied so long as X and Y are real quantities unless $X = 0$, $Y = 0$; therefore,

$$X = \Sigma P \cos \alpha = 0 \text{ and } Y = \Sigma P \sin \alpha = 0. \qquad (2)$$

Hence these are the two necessary and sufficient conditions for the equilibrium of the forces; that is, *the algebraic sum of the rectangular components of the forces, along each of two right lines at right angles to each other, in the plane of the forces, is equal to zero.* As the conditions of equilibrium must be independent of the system of co-ordinate axes, it follows that, if any number of concurring forces in one plane are in equilibrium, *the algebraic sum of the rectangular components of the forces along every right line in their plane is zero.*

EXAMPLES.

1. Given four equal concurring forces whose directions are inclined to the axis of x at angles of 15°, 75°, 135°, and 225°; determine the magnitude and direction of their resultant.

Let each force be equal to P; then

$$X = P \cos 15^\circ + P \cos 75^\circ + P \cos 135^\circ + P \cos 225^\circ$$
$$= P \frac{3^{\frac{1}{2}} - 2}{2^{\frac{1}{2}}}.$$

$$Y = P \sin 15^\circ + P \sin 75^\circ + P \sin 135^\circ + P \sin 225^\circ$$
$$= P \left(\tfrac{3}{2}\right)^{\frac{1}{2}}.$$

$$\therefore \quad R = P \left(5 - 2\sqrt{3}\right)^{\frac{1}{2}}.$$

$$\tan \theta = \frac{3^{\frac{1}{2}}}{3^{\frac{1}{2}} - 2}.$$

2. Given two equal concurring forces, P, whose directions are inclined to the axis of x at angles of 30° and 315°; find their resultant. *Ans.* $R = 1.59\ P$.

3. Given three concurring forces of 4, 5, and 6 lbs., whose directions are inclined to the axis of x at angles of 0°, 60°, and 135° respectively; find their resultant.

Ans. $R = \sqrt{97 + 15\sqrt{6} - 39\sqrt{2}}$.

4. Given three equal concurring forces, P, whose directions are inclined to the axis of x at angles of 30°, 60°, and 165°; find their resultant. *Ans.* $R = 1.67\ P$.

5. Given three concurring forces, 100, 50, and 200 lbs., whose directions are inclined to the axis of x at angles of 0°, 60°, and 180°; find the magnitude and direction of their resultant. *Ans.* $R = 86.6$ lbs.; $\theta = 150°$.

38. To find the Magnitude and Direction of the Resultant of any number of Concurring Forces in Space.—Let P_1, P_2, P_3, etc., be the forces, and the whole be referred to a system of rectangular co-ordinates. Let α_1, β_1, γ_1, be the angles which the direction of P_1 makes with three rectangular axes drawn through the point of application; let α_2, β_2, γ_2, be the angles which the direction of P_2 makes with the same axes; α_3, β_3, γ_3, the angles which P_3 makes with the same axes, etc. Resolve these forces along the co-ordinate axes (Art. 35); the components of P_1 along the axes are $P_1 \cos \alpha_1$, $P_1 \cos \beta_1$, $P_1 \cos \gamma_1$. Resolve each of the other forces in the same way, and let X, Y, Z, be the algebraic sums of the components of the forces along the axes of x, y, and z, respectively; then we have

$$\left.\begin{aligned} X &= P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \text{etc.} \\ &= \Sigma P \cos \alpha. \\ Y &= P_1 \cos \beta_1 + P_2 \cos \beta_2 + P_3 \cos \beta_3 + \text{etc.} \\ &= \Sigma P \cos \beta. \\ Z &= P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + P_3 \cos \gamma_3 + \text{etc.} \\ &= \Sigma P \cos \gamma. \end{aligned}\right\} \quad (1)$$

Let R be the resultant of all the forces; and let the angles which its direction makes with the three axes be a, b, c; then as the resolved parts of R along the three co-ordinate axes are equal to the sum of the resolved parts of the several components along the same axes, we have

$$R \cos a = X, \quad R \cos b = Y, \quad R \cos c = Z. \qquad (2)$$

Squaring, and adding, we get

$$R^2 = X^2 + Y^2 + Z^2; \qquad (3)$$

$$\cos a = \frac{X}{R}, \quad \cos b = \frac{Y}{R}, \quad \cos c = \frac{Z}{R}; \qquad (4)$$

which determines the magnitude of the resultant of any system of forces in space and the angles its direction makes with three rectangular axes.

39. The Conditions of Equilibrium for any number of Concurring Forces in Space.—If the forces are in equilibrium, $R = 0$; therefore (3) of Art. 38 becomes

$$X^2 + Y^2 + Z^2 = 0.$$

But as every square is essentially positive, this cannot be unless $X = 0$, $Y = 0$, $Z = 0$; and therefore

$$\Sigma P \cos \alpha = 0, \quad \Sigma P \cos \beta = 0, \quad \Sigma P \cos \gamma = 0; \qquad (1)$$

and these are the conditions among the forces that they may be in equilibrium; that is, the sum of the components of the forces along each of the three co-ordinate axes is equal to zero.

40. Tension of a String.—By the *tension* of a string is meant the pull along its fibres which, at any point, tends to stretch or break the string. In the application of the preceding principles the *string* or *cord* is often used as a means of communicating force. A string is said to be perfectly *flexible* when any force, however small, which is applied otherwise than along the direction of the string, will change its form. In this work the string will be regarded as perfectly flexible, inextensible, and without weight.

If such a string be kept in equilibrium by two forces, one at each end, it is clear that these forces must be equal and act in opposite directions, so that the string assumes the form of a straight line in the direction of the forces. In this case the *tension* of the string is the same throughout, and is measured by the force applied at one end; and if it passes over a smooth peg, or over any number of smooth surfaces, its tension is the same at all of its points. If the string should be *knotted* at any of its points to other strings, we must regard its continuity as broken, and the tension, in this case, will not be the same in the two portions which start from the knot.

EXAMPLES.

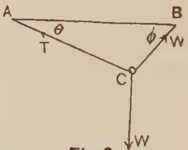

Fig. 8

1. A and B (Fig. 8) are two fixed points in a horizontal line; at A is fastened a string of length b, with a smooth ring at its other extremity, C, through which passes another string with one end fastened at B, the other end of which is attached to a given weight W; it is required to determine the position of C.

Before setting about the solution of statical problems of this kind, the student will clear the ground before him, and greatly simplify his labor by asking himself the following questions: (1) What lines are there in the figure whose lengths are already given? (2) What forces are there whose magnitudes are already given, and what are the forces whose magnitudes are yet unknown? (3) What variable lines or angles in the figure would, if they were known, determine the required position of C ?

Now in this problem, (1) the linear magnitudes which are given are the lines AB and AC. (2) The forces acting at the point C to keep it at rest are the weight W, a tension in the string CB, and another tension in the string CA. Of these W is given, and so is the tension in CB, which must also be equal to W, since the ring is smooth and the tension therefore of WCB is the same throughout and of course equal to W. But as yet there is nothing determined about the magnitude of the tension in CA. And (3) the angle of inclination of the string CA to the horizon would, if known, at once determine the position of C. For if this angle is known, we can draw AC of the given length; then joining C to B, the position of the system is completely known.

Let AB $= a$, AC $= b$, CAB $= \theta$, CBA $= \phi$, and the tension of the string AC $= T$. Then, for the equilibrium of the point C under the action of the three forces, W, W, and T, we apply (2) of Art. 37, and resolve the forces horizontally and vertically; and equate those acting towards the right-hand to those acting towards the left; and those acting upwards to those acting downwards. Then the horizontal and vertical forces are respectively

$$W \cos \phi = T \cos \theta;$$

$$W \sin \phi + T \sin \theta = W.$$

Eliminating T we have

$$\cos \theta = \sin (\theta + \phi);$$

$$\therefore \quad 2\theta + \phi = 90^\circ. \tag{1}$$

Also, from trigonometry we have

$$\frac{\sin (\theta + \phi)}{\sin \phi} = \frac{a}{b}; \tag{2}$$

from (1) and (2) θ and ϕ may be found; and therefore T may be found; and thus all the circumstances of the problem are determined.

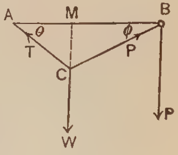

Fig. 9

2. One end of a string is attached to a fixed point, A, (Fig. 9); the string, after passing over a smooth peg, B, sustains a given weight, P, at its other extremity, and to a given point, C, in the string is knotted a given weight, W. Find the position of equilibrium.

The entire length of the string, ACBP, is of no consequence, since it is clear that, once equilibrium is established, P might be suspended from a point at any distance whatever from B. The forces acting at the point, C, are the given weight, W, the tension in the string, CB, which, since the peg is smooth, is P, and the tension in the string CA, which is unknown.

Let AB $= a$, AC $= b$, CAB $= \theta$, CBA $= \phi$, and the tension of the string, AC $= T$. Then for the equilibrium of the point C, we have (Art. 32),

$$\frac{P}{W} = \frac{\cos\theta}{\sin(\theta+\phi)}; \tag{1}$$

also, from the geometry of the figure, we have

$$b\sin(\theta+\phi) = a\sin\phi. \tag{2}$$

From (1) and (2) we get

$$\frac{P}{W} = \frac{b\cos\theta}{a\sin\phi},$$

or

$$\sin\phi = \frac{bW}{aP}\cos\theta;$$

$$\therefore \cos\phi = \frac{\sqrt{a^2P^2 - b^2W^2\cos^2\theta}}{aP}.$$

Expanding $\sin(\theta + \phi)$ in (2), and substituting in it these values of $\sin\phi$ and $\cos\phi$, and reducing, we have the equation

$$\cos^3\theta - \frac{P^2a^2 + W^2(a^2+b^2)}{2abW^2}\cos^2\theta + \frac{P^2a}{2W^2b} = 0,$$

from which θ may be found. (See Minchin's Statics, p. 29.)

3. If, in the last example, the weight, W, instead of being knotted to the string at C, is suspended from a smooth ring which is at liberty to slide along the string, ACB, find the position of equilibrium.

Ans. $\sin\theta = \frac{W}{2P}$.

41. Equilibrium of Concurring Forces on a Smooth Plane.—If a particle be kept at rest on a smooth surface, plane or curved, by the action of any number of forces applied to it, the resultant of these forces must be in the direction of the normal to the surface at the point where the particle is situated, and must be equivalent to the pressure which the surface sustains. For, if the resultant had any other direction it could be resolved into two components, one in the direction of the normal and the other in the direction of a tangent; the first of these would be opposed by the reaction of the surface; the second being unopposed, would cause the particle to move. Hence, we may dispense with the plane altogether, and regard its normal reaction as one of the forces by which the particle is kept at rest. Therefore if the particle on which the statical forces act be on a smooth plane surface, the case is the same as that treated in Art. 39, viz., equilibrium of a particle acted upon by any number of forces; and in writing down the equations of equilibrium, we merely have to include the normal reaction of the plane among all the others.

EXAMPLES.

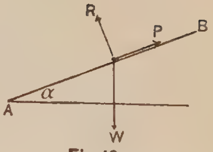

Fig. 10

1. A heavy particle is placed on a smooth inclined plane, AB, (Fig. 10), and is sustained by a force, P, which acts along AB in the vertical plane which is at right angles to AB; find P, and also the pressure on the inclined plane.

The only effect of the inclined plane is to produce a normal reaction, R, on the particle. Hence if we introduce this force, we may imagine the plane removed.

Let W be the weight of the particle, and α the inclination of the plane to the horizon.

Resolving the forces along, and perpendicular to AB, since the lines along which forces may be resolved are arbitrary (Art. 37), we have successively,

$$P - W \sin \alpha = 0, \quad \text{or} \quad P = W \sin \alpha;$$

and $$R - W \cos \alpha = 0, \quad \text{or} \quad R = W \cos \alpha.$$

If, for example, the weight of the particle is 4 oz., and the inclination of the plane 30°, there will be a normal pressure of $2\sqrt{3}$ oz. on the plane, and the force, P, will be 2 oz.

2. In the previous example, if P act horizontally, find its magnitude, and also that of R.

Resolving along AB and perpendicular to it, we have successively,

$$P \cos \alpha - W \sin \alpha = 0, \quad \text{or} \quad P = W \tan \alpha;$$

and $$P \sin \alpha + W \cos \alpha - R = 0, \quad \therefore \quad R = \frac{W}{\cos \alpha}.$$

3. If the particle is sustained by a force, P, making a given angle, θ, with the inclined plane, find the magnitude of this force, and of the pressure on the plane, all the forces acting in the same vertical plane.

Resolving along and perpendicular to the plane successively, we have

$$P \cos \theta - W \sin \alpha = 0,$$

and $$R + P \sin \theta - W \cos \alpha = 0,$$

from which we obtain

$$P = W \frac{\sin \alpha}{\cos \theta}; \quad R = W \frac{\cos (\alpha + \theta)}{\cos \theta}.$$

Rem.—The advantage of a judicious selection of directions for the resolution of the forces is evident. By resolving at right angles to one of the unknown forces, we obtain an equation free from that force; whereas if the directions are selected at random, all of the forces will enter each equation, which will make the solution less simple.

The student will observe that these values of P and R could have been obtained at once, without resolution, by Art. 32.

42. Conditions of Equilibrium for any number of Concurring Forces when the particle on which they act is Constrained to Remain on a Given Smooth Surface.—If a particle be kept at rest on a smooth surface by the action of any number of forces applied to it, the resultant of these forces must be in the direction of the normal to the surface at the point where the particle is situated, and must be equivalent to the pressure which the surface sustains (Art. 41). Hence since the resultant is in the direction of the normal, and is destroyed by the reaction of the surface, we may regard this reaction as an additional force directly opposed to the normal force.

Let N be the normal reaction of the surface, and α, β, γ, the angles which N makes with the co-ordinate axes of x, y, and z, respectively. Let X, Y, Z, be the sum of the components of all the other forces resolved parallel to the three axes respectively. The reaction N may be considered a new force, which, with the other forces, keeps the particle in equilibrium. Therefore, resolving N parallel to the three axes, we have (Art. 39),

$$\left.\begin{aligned} X + N\cos\alpha &= 0, \\ Y + N\cos\beta &= 0, \\ Z + N\cos\gamma &= 0. \end{aligned}\right\} \qquad (1)$$

Let $u = f(x, y, z) = 0$, be the equation of the given surface, and x, y, z the co-ordinates of the particle to which the forces are applied. We have (Anal. Geom., Art. 175),

$$\left.\begin{aligned} \cos\alpha &= \frac{a'}{\sqrt{a'^2 + b'^2 + 1}}, \\ \cos\beta &= \frac{b'}{\sqrt{a'^2 + b'^2 + 1}}, \\ \cos\gamma &= \frac{1}{\sqrt{a'^2 + b'^2 + 1}}, \end{aligned}\right\} \qquad (2)$$

where a' and b' are the tangents of the angles which the projections of the normal, N, on the co-ordinate planes xz and yz make with the axis of z. Since the normal is perpendicular to the plane tangent to the surface at (x, y, z), the projections of the normal are perpendicular to the traces of the plane. Therefore (Anal. Geom., Art. 27, Cor. 1), we have

$$1 + aa' = 0, \qquad (3)$$

and

$$1 + bb' = 0; \qquad (4)$$

in which

$$a = \frac{dx}{dz}, \quad a' = \frac{dx'}{dz'}, \quad b = \frac{dy}{dz}, \quad b' = \frac{dy'}{dz'},$$

(Calculus, Art. 56*a*.) Substituting in (3) and (4), we have

$$1 + \frac{dx}{dz} \cdot \frac{dx'}{dz'} = 0;$$

and
$$1 + \frac{dy}{dz} \cdot \frac{dy'}{dz'} = 0;$$

from which

$$\frac{dx'}{dz'} = -\frac{dz}{dx} = \frac{\frac{du}{dx}}{\frac{du}{dz}} \text{ (Cal. Art. 87)} = a', \tag{5}$$

and
$$\frac{dy'}{dz'} = -\frac{dz}{dy} = \frac{\frac{du}{dy}}{\frac{du}{dz}} = b'. \tag{6}$$

Substituting these values of a' and b' in (2) and multiplying both terms of the fraction by $\frac{du}{dz}$, we have

$$\left.\begin{aligned} \cos\alpha &= \frac{\frac{du}{dx}}{\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}}, \\ \cos\beta &= \frac{\frac{du}{dy}}{\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}}, \\ \cos\gamma &= \frac{\frac{du}{dz}}{\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}}, \end{aligned}\right\} \tag{7}$$

which give the values of the direction cosines of the normal at (x, y, z).

Putting the denominator equal to Q, for shortness, and substituting in (1) and transposing, we have

$$X = -\frac{N}{Q} \cdot \frac{du}{dx}, \tag{8}$$

$$Y = -\frac{N}{Q} \cdot \frac{du}{dy}, \tag{9}$$

$$Z = -\frac{N}{Q} \cdot \frac{du}{dz} \tag{10}$$

Eliminating N between these three equations, we obtain the two independent equations,

$$\frac{X}{\frac{du}{dx}} = \frac{Y}{\frac{du}{dy}} = \frac{Z}{\frac{du}{dz}}, \tag{11}$$

which express the conditions that must exist among the applied forces and their directions in order that their resultant may be normal to the surface, *i. e.*, that there may be equilibrium. If these two equations are not satisfied, equilibrium on the surface cannot exist. Hence the point on a given surface, at which a given particle under the action of given forces will rest in equilibrium, is the point at which equations (11) are satisfied.

COR. 1.—Squaring equations (8), (9), (10) and adding, we get

$$X^2 + Y^2 + Z^2 = N^2\left[\frac{\left(\frac{du}{dx}\right)^2}{Q^2} + \frac{\left(\frac{du}{dy}\right)^2}{Q^2} + \frac{\left(\frac{du}{dz}\right)^2}{Q^2}\right] = N^2;$$

$$\therefore \quad N = \sqrt{X^2 + Y^2 + Z^2}, \tag{12}$$

which is the value of the normal resistance of the surface and is precisely the same as the resultant of the acting forces, as it clearly should be; but this resistance must act in the direction opposite to that of the resultant.

Cor. 2.—Multiplying (8), (9), (10) by dx, dy, dz, respectively, and adding, and remembering that the total differential of $u = 0$ is zero, we get

$$Xdx + Ydy + Zdz = 0, \tag{13}$$

which is an equation of condition for equilibrium. If (13) cannot be satisfied at any point of the surface, equilibrium is impossible.

Cor. 3.—If the forces all act in one plane, the surface becomes a plane curve; let this curve be in the plane xy, then $z = 0$; therefore (11) and (13) become

$$\frac{X}{\frac{du}{dx}} = \frac{Y}{\frac{du}{dy}}, \tag{14}$$

and

$$Xdx + Ydy = 0, \tag{15}$$

in which (14) or (15) may be used according as the equation of the curve is given as an implicit or explicit function.

EXAMPLES.

1. A particle is placed on the surface of an ellipsoid, and is acted on by attracting forces which vary directly as the distance of the particle from the principal planes* of section; it is required to determine the position of equilibrium.

Let the equation of the ellipsoid be

$$u = f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0;$$

---

* Planes of xy, yz, zx.

$$\therefore \quad \frac{du}{dx} = \frac{2x}{a^2}, \quad \frac{du}{dy} = \frac{2y}{b^2}, \quad \frac{du}{dz} = \frac{2z}{c^2};$$

and let the x-, y-, and z-components of the forces be respectively,

$$X = -u_1 x, \quad Y = -u_2 y, \quad Z = -u_3 z;$$

then (11) will give

$$u_1 a^2 = u_2 b^2 = u_3 c^2;$$

which may be put in the form

$$\frac{u_1}{a^{-2}} = \frac{u_2}{b^{-2}} = \frac{u_3}{c^{-2}} = \frac{u_1 + u_2 + u_3}{a^{-2} + b^{-2} + c^{-2}}.$$

If these conditions are fulfilled, the particle will rest at all points of the surface.

2. Again, take the same surface, and let the forces vary inversely as the distances of the point from the principal planes; it is required to determine the position of equilibrium.

Here $X = -\frac{u_1}{x}, \quad Y = -\frac{u_2}{y}, \quad Z = -\frac{u_3}{z};$

therefore (11) becomes

$$\frac{\frac{x^2}{a^2}}{u_1} = \frac{\frac{y^2}{b^2}}{u_2} = \frac{\frac{z^2}{c^2}}{u_3} = \frac{1}{u_1 + u_2 + u_3} = \frac{1}{u}$$

by putting u for $u_1 + u_2 + u_3$,

$$\therefore \quad x = a\left(\frac{u_1}{u}\right)^{\frac{1}{2}}, \quad y = b\left(\frac{u_2}{u}\right)^{\frac{1}{2}}, \quad z = c\left(\frac{u_3}{u}\right)^{\frac{1}{2}},$$

which in (12) gives

$$N^2 = \frac{u_1^{\ 2}}{x^2} + \frac{u_2^{\ 2}}{y^2} + \frac{u_3^{\ 2}}{z^2}$$

$$= u \left[\frac{u_1}{a^2} + \frac{u_2}{b^2} + \frac{u_3}{c^2} \right].$$

3. A particle is placed inside a smooth sphere on the concave surface, and is acted on by gravity and by a repulsive force which varies inversely as the square of the distance from the lowest point of the sphere; find the position of equilibrium of the particle.

Let the lowest point of the sphere be taken for the origin of co-ordinates, and let the axis of z be vertical, and positive upwards; then the equation of the sphere, whose radius is a, is

$$x^2 + y^2 + z^2 - 2az = 0.$$

Let $W =$ the weight of the particle, and $r =$ the distance of it from the lowest point; then

$$r^2 = x^2 + y^2 + z^2 = 2az.$$

Also, let the repulsive force at the unit's distance $= u$; then at the distance r it will be

$$= \frac{u}{r^2} = \frac{u}{2az};$$

$$\therefore \quad X = \frac{u}{2az} \cdot \frac{x}{r},$$

$$Y = \frac{u}{2az} \cdot \frac{y}{r},$$

$$Z = \frac{u}{2az} \cdot \frac{z}{r} - W.$$

Let $N =$ the normal pressure of the curve; then (8) and (10) give

$$\frac{u}{2az} \cdot \frac{x}{r} + N\frac{x}{a} = 0,$$

$$\frac{u}{2az} \cdot \frac{z}{r} - W + N\frac{z-a}{a} = 0;$$

from which we have

$$r^3 = \frac{ua}{W}; \quad z = \frac{u^{\frac{2}{3}}}{2a^{\frac{1}{3}}W^{\frac{2}{3}}};$$

whence the position of the particle is known for a given weight, and for a given value of u. (See Price's Anal. Mechanics, Vol. I, p. 39.)

4. Two weights, P and Q, are fastened to the ends of a string, (Fig. 11), which passes over a pulley, O; and Q hangs freely when P rests on a plane curve, AP, in a vertical plane; it is required to find the position of equilibrium when the curve is given.

The forces which act on P are (1) the tension of the string in the line OP, which is equal to the weight of Q, (2) the weight of P acting vertically downwards, (3) the normal reaction of the curve R.

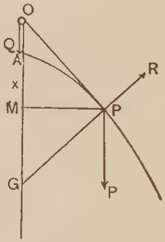

Fig. 11

Let O be the origin of co-ordinates, and the axis of x vertical and positive downwards. Let $OM = x$, $MP = y$, $OP = r$, $POM = \theta$, $OA = a$. Then,

$$X = P - Q\cos\theta - R\frac{dy}{ds},$$

$$Y = -Q\sin\theta + R\frac{dx}{ds};$$

therefore from (15) we have

$$(P - Q \cos \theta)\, dx - Q \sin \theta\, dy = 0,$$

or

$$Pdx - Q\frac{x\, dx + y\, dy}{r} = 0.$$

But since $x^2 + y^2 = r^2$,

we have $x\, dx + y\, dy = r\, dr$;

$$\therefore \quad Pdx - Qdr = 0; \tag{1}$$

which is the condition that must be satisfied by P, Q, and the equation of the curve.

5. Required the equation of the curve, on all points of which P will rest.

Integrating (1) of Ex. 4, we have

$$Px - Qr = C. \tag{1}$$

But since P is to rest at all points of the curve, this equation must be satisfied when P is at A, from which we get $x = r = a$; therefore (1) becomes

$$Pa - Qa = C;$$

which in (1) gives

$$r = \frac{\left(1 - \frac{P}{Q}\right) a}{1 - \frac{P}{Q} \cos \theta};$$

which is the equation of a conic section, of which the focus is at the pole O; and is an ellipse, parabola, or hyperbola, according as $P <, =,$ or $> Q$.

EXAMPLES.

1. Two forces of 10 and 20 lbs. act on a particle at an angle of 60°; find the resultant. *Ans.* 26.5 lbs.

2. The resultant of two forces is 10 lbs.; one of the forces is 8 lbs., and the other is inclined to the resultant at an angle of 36°. Find it, and also find the angle between the two forces. (There are two solutions, this being the ambiguous case in the solution of a triangle.)

Ans. Force is 2.66 lbs., or 13.52 lbs. Angle is 47° 17′ 05″, or 132° 42′ 55″.

3. A point is kept at rest by forces of 6, 8, 11 lbs. Find the angle between the forces 6 and 8.

Ans. 77° 21′ 52″.

4. The directions of two forces acting at a point are inclined to each other (1) at an angle of 60°, (2) at an angle of 120°, and the respective resultants are as $\sqrt{7} : \sqrt{3}$; compare the magnitude of the forces.

Ans. 2 : 1.

5. Three posts are placed in the ground so as to form an equilateral triangle, and an elastic string is stretched round them, the tension of which is 6 lbs.; find the pressure on each post. *Ans.* $6\sqrt{3}$.

6. The angle between two unknown forces is 37°, and their resultant divides this angle into 31° and 6°; find the ratio of the component forces. *Ans.* 4.927 : 1.

7. If two equal rafters support a weight, W, at their upper ends, required the compression on each. Let the length of each rafter be a, and the horizontal distance between their lower ends be b.

Ans. $\dfrac{aW}{\sqrt{4a^2 - b^2}}$.

8. Three forces act at a point, and include angles of 90° and 45°. The first two forces are each equal to $2P$, and the resultant of them all is $\sqrt{10}P$; find the third force.

Ans. $P\sqrt{2}$.

9. Find the magnitude, R, and direction, θ, of the resultant of the three forces, $P_1 = 30$ lbs., $P_2 = 70$ lbs., $P_3 = 50$ lbs., the angle included between P_1 and P_2 being 56°, and between P_2 and P_3 104°. (It is generally convenient to take the action line of one of the forces for the axis of x.)

Let the axis of x coincide with the direction of P_1; then (Art. 36), we have

$$X = 22.16; \quad Y = 75.13; \quad R = 78.33; \quad \theta = 73° \, 34'.$$

10. Three forces of 10 lbs. each act at the same point; the second makes an angle of 30° with the first, and the third makes an angle of 60° with the second; find the magnitude of the resultant. *Ans.* 24 lbs., nearly.

11. If three forces of 99, 100, and 101 units respectively, act on a point at angles of 120°; find the magnitude of their resultant, and its inclination to the force of 100.

Ans. $\sqrt{3}$; 90°.

12. A block of 800 lbs. is so situated that it receives from the water a pressure of 400 lbs. in a south direction, and a pressure from the wind of 100 lbs. in a westerly direction; required the magnitude of the resultant pressure, and its direction with the vertical.

Ans. 900 lbs.; 27° 16′.

13. A weight of 40 lbs. is supported by two strings, one of which makes an angle of 30° with the vertical, the other 45°; find the tension in each string.

Ans. $20(\sqrt{6} - \sqrt{2})$; $40(\sqrt{3} - 1)$.

14. Two forces, P and P', acting along the diagonals of a parallelogram, keep it at rest in such a position that one of its sides is horizontal; show that

$$P \sec \alpha' = P' \sec \alpha = W \operatorname{cosec} (\alpha + \alpha'),$$

where W is the weight of the parallelogram, and α and α' the angles between the diagonals and the horizontal side.

15. Two persons pull a heavy weight by ropes inclined to the horizon at angles of 60° and 30° with forces of 160 lbs. and 200 lbs. The angle between the two vertical planes of the ropes is 30°; find the single horizontal force that would produce the same effect. *Ans.* 245.8 lbs.

16. In order to raise vertically a heavy weight by means of a rope passing over a fixed pulley, three workmen pull at the end of the rope with forces of 40 lbs., 50 lbs., and 100 lbs.; the directions of these forces being inclined to the horizon at an angle of 60°. What is the magnitude of the resultant force which tends directly to raise the weight?

Ans. 164.54 lbs.

17. Three persons pull a heavy weight by cords inclined to the horizon at an angle of 60°, with forces of 100, 120, and 140 lbs. The three vertical planes of the cords are inclined to each other at angles of 30°; find the single horizontal force that would produce the same effect.

Ans. $10 \sqrt{145 + 72 \sqrt{3}}$ lbs.

18. Two forces, P and Q, acting respectively parallel to the base and length of an inclined plane, will each singly sustain on it a particle of weight, W; to determine the weight of W.

Let α = inclination of the plane to the horizon; then resolving in each case along the plane, so that the normal pressures may not enter into the equations (See Rem., Ex. 3, Art. 41), we have

$$P \cos \alpha = W \sin \alpha ; \quad Q = W \sin \alpha ;$$

$$\therefore \quad W = \frac{PQ}{(P^2 - Q^2)^{\frac{1}{2}}}.$$

19. A cord whose length is $2l$, is fastened at A and B, in the same horizontal line, at a distance from each other equal to $2a$; and a smooth ring upon the cord sustains a weight W; find the tension of the cord.

Ans. $T = \dfrac{Wl}{2\sqrt{l^2 - a^2}}$.

20. A heavy particle, whose weight is W, is sustained on a smooth inclined plane by three forces applied to it, each equal to $\dfrac{W}{3}$; one acts vertically upward, another horizontally, and the third along the plane; find the inclination, α, of the plane.

Ans. $\tan \dfrac{\alpha}{2} = \dfrac{1}{2}$.

21. A body whose weight is 10 lbs. is supported on a smooth inclined plane by a force of 2 lbs. acting along the plane, and a horizontal force of 5 lbs. Find the inclination of the plane. *Ans.* $\sin^{-1} \frac{3}{5}$.

22. A body is sustained on a smooth inclined plane (inclination α) by a force, P, acting along the plane, and a horizontal force, Q. When the inclination is halved, and the forces, P and Q, each halved, the body is still observed to rest; find the ratio of P to Q.

Ans. $\dfrac{P}{Q} = 2 \cos^2 \dfrac{\alpha}{4}$.

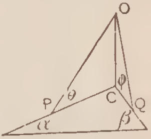

Fig. 12

23. Two weights, P and Q, (Fig. 12), rest on a smooth double-inclined plane, and are attached to the extremities of a string which passes over a smooth peg, O, at a point vertically over the intersection of the planes, the peg and the weights being in a

vertical plane. Find the position of equilibrium, if $l =$ the length of the string and $h =$ CO.

Ans. The position of equilibrium is given by the equations

$$P\frac{\sin\alpha}{\cos\theta} = Q\frac{\sin\beta}{\cos\phi},$$

$$\frac{\cos\alpha}{\sin\theta} + \frac{\cos\beta}{\sin\phi} = \frac{l}{h}.$$

24. Two weights, P and Q, connected by a string, length l, rest on the convex side of a smooth vertical circle, radius a. Find the position of equilibrium, and show that the heavier weight will be higher up on the circle than the lighter, the radius of the circle drawn to P making an angle θ with the vertical diameter.

Ans. $P\sin\theta = Q\sin\left(\frac{l}{a} - \theta\right)$.

25. Two weights, P and Q, connected directly by a string of given length, rest on the convex side of a smooth vertical circle, the string forming a chord of the circle; find the position of equilibrium.

Ans. If 2α is the angle subtended at the centre of the circle by the string, the inclination, θ, of the string to the vertical is given by the equation

$$\cot\theta = \frac{P-Q}{P+Q}\tan\alpha.$$

26. Two weights, P and Q, (Fig. 13), rest on the concave side of a parabola whose axis is horizontal, and are connected by a string, length l, which passes over a smooth peg at the focus, F. Find the position of equilibrium.

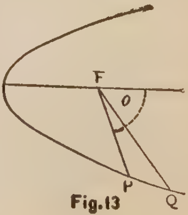

Fig. 13

Ans. Let $\theta =$ the angle which FP

makes with the axis, and $4m$ = the latus rectum of the parabola, then

$$\cot\frac{\theta}{2} = \frac{P\sqrt{l - 2m}}{\sqrt{m(P^2 + Q^2)}}.$$

27. A particle is placed on the convex side of a smooth ellipse, and is acted upon by two forces, F and F', towards the foci, and a force, F'', towards the centre. Find the position of equilibrium.

Ans. $r = \frac{b}{\sqrt{1 - n^2}}$, where r = the distance of the particle from the centre of the ellipse; b = semi-minor axis, and $n = \frac{F - F'}{F''}$.

28. Let the curve, (Fig. 11), be a circle in which the origin and pulley are at a distance, a, above the centre of the circle; to determine the position of equilibrium.

Ans. $r = \frac{Q}{P}a$.

29. Let the curve, (Fig. 11), be a hyperbola in which the origin and pulley are at the centre, O, the transverse axis being vertical; to determine the position of equilibrium.

Ans. $x = \frac{bP}{l(P^2 - l^2Q^2)^{\frac{1}{2}}}$.

30. A particle, P, is acted upon by two forces towards two fixed points, S and H, these forces being $\frac{\mu}{SP}$ and $\frac{\mu}{HP}$, respectively; prove that P will rest at all points inside a smooth tube in the form of a curve whose equation is $SP.PH = k^2$, k being a constant.

31. Two weights, P and Q, connected by a string, rest on the convex side of a smooth cycloid. Find the position of equilibrium.

Ans. If l = the length of the string, and a = radius of generating circle, the position of equilibrium is defined by the equation

$$\sin\frac{\theta}{2} = \frac{Q}{P+Q}\cdot\frac{l}{4a},$$

where θ is the angle between the vertical and the radius to the point on the generating circle which corresponds to P.

32. Two weights, P and Q, rest on the convex side of a smooth vertical circle, and are connected by a string which passes over a smooth peg vertically over the centre of the circle; find the position of equilibrium.

Ans. Let h = the distance between the peg, B, and the centre of the circle; θ and ϕ = the angles made with the vertical by the radii to P and Q, respectively; α and β = the angles made with the tangents to the circle at P and Q by the portions PB and QB of the string; l = length of the string; then

$$P\frac{\sin\theta}{\cos\alpha} = Q\frac{\sin\phi}{\cos\beta},$$

$$h\left(\frac{\sin\theta}{\cos\alpha} + \frac{\sin\phi}{\cos\beta}\right) = l,$$

$$h\cos(\theta+\alpha) = a\cos\alpha,$$

$$h\cos(\phi+\beta) = a\cos\beta.$$

CHAPTER III.

COMPOSITION AND RESOLUTION OF FORCES ACTING ON A RIGID BODY.

43. A Rigid Body.—In the last chapter we considered the action of forces which have a common point of application. We shall now consider the action of forces which are applied at different points of a rigid body.

A *rigid body* is one in which the particles retain invariable positions with respect to one another, so that no external force can alter them. Now, as a matter of fact, there is no such thing in nature as a body that is perfectly rigid; every body yields more or less to the forces which act on it. If, then, in any case, the body is altered or compressed appreciably, we shall suppose that it has assumed its figure of equilibrium, and then consider the points of application of the forces as a system of invariable form. The term *body* in this work means *rigid* body.

44. Transmissibility of Force.—When a force acts at a definite point of a body and along a definite line, the effect of the force will be unchanged at whatever point of its direction we suppose it applied, provided this point be either one of the points of the body, or be invariably connected with the body. This principle is called the *transmissibility of a force to any point in its line of action.*

Now two equal forces acting on a particle in the same line and in opposite directions neutralize each other (Art. 16); so by this principle two equal forces acting in the same line and in opposite directions at any points of a rigid body in that line neutralize each other. Hence it is clear that when many forces are acting on a rigid body, any two, which are equal and have the same line of action and act in opposite directions, may be omitted, and also that two equal forces along the same line of action and in opposite directions, may be introduced without changing the circumstances of the system.

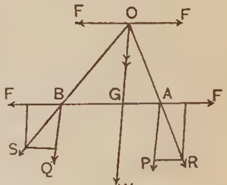

Fig. 14

45. Resultant of Two Parallel Forces.*—(1) Let P and Q, (Fig. 14), be the two parallel forces acting at the points A and B, in the same direction, on a rigid body. It is required to find the resultant of P and Q.

At A and B introduce two equal and opposite forces, F. The introduction of these forces will not disturb the action of P and Q (Art. 44). P and F at A are equivalent to a single force, R, and Q and F at B are equivalent to a single force, S. Then let R and S be supposed to act at O, the point of intersection of their lines of action. At this point let them be resolved into their components, P, F, and Q, F, respectively. The two forces, F, at O, neutralize each other, while the components, P and Q, act in the line OG, parallel to their lines of action at A and B. Hence the *magnitude* of the resultant is $P + Q$, (Art. 28). To find the point, G, in which its line of action cuts AB, let the extremities of P and R (acting at A) be joined, and complete the parallelogram. Then the triangle PAR is evidently similar to GOA; therefore,

$$\frac{P}{F} = \frac{\text{GO}}{\text{GA}}; \text{ similarly } \frac{Q}{F} = \frac{\text{GO}}{\text{GB}};$$

therefore, by division,

$$\frac{P}{Q} = \frac{\text{GB}}{\text{GA}}. \qquad (1)$$

---

* Minchin's Statics, p. 85.

(2) *When the forces act in opposite directions.*—At A and B, (Fig 15), apply two equal and opposite forces F, as before, and let R, the resultant of P and F, and S, the resultant of Q and F, be transferred to O, their point of intersection. If at O the forces, R and S, are decomposed into their original components, the two forces, F, destroy each other, the force, P, will act in the direction GO parallel to the direction of P and Q, and the force Q will act in the direction OG. Hence the resultant is a force $= P - Q$, acting in the line GO. To find the point G, we have, from the similar triangles, PAR and OGA,

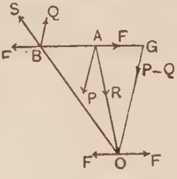

Fig. 15

$$\frac{P}{F} = \frac{\mathrm{GO}}{\mathrm{GA}}; \text{ also } \frac{Q}{F} = \frac{\mathrm{GO}}{\mathrm{GB}};$$

$$\therefore \quad \frac{P}{Q} = \frac{\mathrm{GB}}{\mathrm{GA}}. \tag{2}$$

Hence *the resultant of two parallel forces, acting in the same or opposite directions, at the extremities of a rigid right line, is parallel to the components, equal to their algebraic sum, and divides the line or the line produced, into two segments which are inversely as the forces.*

In both cases we have the equation

$$P \times \mathrm{GA} = Q \times \mathrm{GB}. \tag{3}$$

Hence the following theorem:

If from a point on the resultant of two parallel forces a right line be drawn meeting the forces, whether perpendicularly or not, the products obtained by multiplying each force by its distance from the resultant, measured along the arbitrary line, are equal.

Sch.—The point G possesses this remarkable property;

that, however P and Q are turned about their points of application, A and B, their directions remaining parallel, G, determined as above, remains fixed. This point is in consequence called the *centre* of the parallel forces, P and Q.

46. Moment of a Force.—*The moment of a force with respect to a point is the product of the force and the perpendicular let fall on its line of action from the point.* The moment of a force measures its tendency to produce rotation about a fixed point or fixed axis. Thus let a force, P, (Fig. 16), act on a rigid body in the plane of the paper, and let an axis perpendicular to this plane pass through the body at any point, O. It is clear that the effect of the force will be to turn the body round this axis (the axis being supposed to be fixed), and the turning effect will depend on the *magnitude of the force, P, and the perpendicular distance, p,* of P from O. If P passes through O, it is evident that no rotation of the body round O can take place, whatever be the magnitude of P; while if P vanishes, no rotation will take place however great p may be. Hence, the measure of the power of the force to produce rotation may be represented by the product

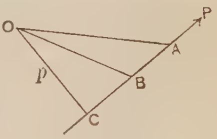

Fig. 16

$$P \cdot p,$$

and this product has received the special name of *Moment.*

The unit of force being a pound and the unit of length a foot, the unit of *moment* will evidently be a *foot-pound.*

The point O is called the *origin of moments*, and may or may not be chosen to coincide with the origin of coordinates. The solution of problems is often greatly simplified by a proper selection of the origin of moments. The perpendicular from the origin of moments to the action line of the force is called the *arm* of the force.

47. Signs of Moments.—A force may tend to turn a body about a point or about an axis, in either of two directions; if one be regarded as *positive* the other must be *negative:* and hence we distinguish between *positive* and *negative* moments. For the sake of uniformity the moment of a force is said to be *negative* when it tends to turn a body from left to right, *i. e.*, in the direction in which the hands of a clock move; and *positive* when it tends to turn the body from right to left, or opposite the direction in which the hands of a clock move.

48. Geometric Representation of the Moment of a Force with respect to a Point.—Let the line AB (Fig. 16), represent the force, P, in magnitude and direction, and p the perpendicular OC; then the moment of P with respect to O is $AB \times p$ (Art. 46). But this is double the area of the triangle AOB. Hence, *the moment of a force with respect to a point is geometrically represented by double the area of the triangle whose base is the line representing the force in magnitude and direction, and whose vertex is the given point.*

49. Case of Two Equal and Opposite Parallel Forces.—If the forces, P and Q, in Art. 45, (Fig. 15) are equal, the equation

$$P \times \mathrm{GA} = Q \times \mathrm{GB}$$

gives $\mathrm{GA} = \mathrm{GB}$, which is true only when G is at infinity on AB; also the resultant, $P - Q$, is equal to zero. Such a system is called a *Couple.*

A Couple consists of two equal and opposite parallel forces acting on a rigid body at a finite distance from each other.

We shall investigate the laws of the composition and resolution of couples, since to these the composition and resolution of forces of every kind acting on a rigid body may be reduced.

50. Moment of a Couple.—Let O (Fig. 17) be any point in the plane of the couple; let fall the perpendiculars Oa and Ob on the action lines of the forces P. Then if O is inside the lines of action of the forces, both forces tend to produce rotation round O in the same direction, and therefore the sum of their moments is equal to

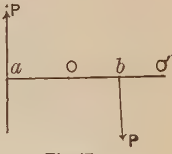

Fig. 17

$$P\,(\mathrm{O}a + \mathrm{O}b), \text{ or } P \times ab$$

If the point chosen is O′, the sum of the moments is evidently

$$P\,(\mathrm{O}'a - \mathrm{O}'b), \text{ or } P \times ab,$$

which is the same as before. Hence the moment of the couple with respect to all points in its plane is constant.

The Arm of a couple is the perpendicular distance between the two forces of the couple.

The Moment of a couple is the product of the arm and one of the forces.

The Axis of a couple is a right line drawn from any chosen point perpendicular to the plane of the couple, and of such length as to represent the magnitude of the moment, and in such direction as to indicate the direction in which the couple tends to turn.

As the motion, in Statics is only *virtual*, and not *actual*, the *direction* of the axis is fixed, but not the *position* of it; it may be any line perpendicular to the plane of the couple, and may be drawn as follows; imagine a watch placed in the plane in which several couples act. Then let the axes of those couples which tend to produce rotation in the direction of the motion of the hands be drawn *downward* through the back of the watch, and the axes of those which tend to produce the contrary rotation be drawn *upward* through the face of the watch. Thus each couple is completely represented by its axis, which is drawn upward or downward according as the moment of the couple is positive or negative; and couples are to be resolved and compounded by the same geometric constructions performed with reference to their axes as forces or velocities, with reference to the lines which directly represent them.

We shall now give three propositions showing that the effect of a couple is not altered when certain changes are made with respect to the couple.

51. *The Effect of a Couple on a Rigid Body is not altered if the arm be turned through any angle about one extremity in the plane of the Couple.*

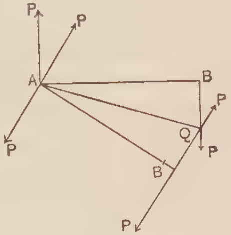

Fig. 18

Let the plane of the paper be the plane of the couple, AB the arm of the original couple, AB′ its new position, and P, P, the forces. At A and B′ respectively introduce two forces each equal to P, with their action lines perpendicular to the arm AB′, and opposite in direction to each other. The effect of the given couple is, of course, unaltered by the introduction of these forces. Let BAB′ $= 2\theta$; then the resultant of P acting at B, and of P acting at B′, whose lines of action meet at Q, is $2P \sin\theta$, acting along the bisector AQ; and the resultant of P acting at A perpendicular to AB and of P perpendicular to AB′, is $2P \sin\theta$, acting along the bisector AQ in a direction opposite to the former resultant. Hence these two resultants neutralize each other; and there remains the couple whose arm is AB′, and whose forces are P, P. Hence the effect of the couple is not altered.

52. *The Effect of a Couple on a Rigid Body is not altered if we transfer the Couple to any other Parallel Plane, the Arm remaining parallel to itself.*

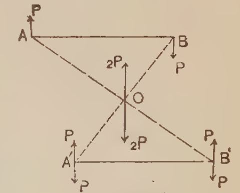

Fig. 19

Let AB be the arm, and P, P, the forces of the given couple; let A′B′ be the new position of the arm parallel to AB. At A′ and B′ apply two equal and opposite forces each equal to P, acting perpendicular to A′B′, and in a plane parallel to the plane of the original couple. This will not alter the effect of the given couple. Join AB′, A′B, bisecting each other at O; then P at A and P at B′, acting in parallel lines, and in the same direction, are equivalent to $2P$ acting at O; also P at B and P at A′, acting in parallel lines and in the same direction, are equivalent to $2P$ acting at O. At O therefore these two resultants, being equal and opposite, neutralize each other; and there remains the couple whose arm is A′B′, and whose forces are each P, acting in the same directions as those of the original couple. Hence the effect of the couple is not altered.

53. *The Effect of a Couple on a Rigid Body is not altered if we replace it by another Couple of which the Moment is the same; the Plane remaining the same and the Arms being in the same straight line and having a common extremity.*

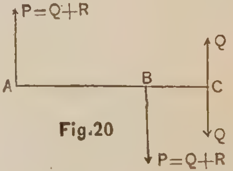

Fig. 20

Let AB be the arm, and P, P, the forces of the given couple, and suppose $P = Q + R$. Produce AB to C so that

$$\text{AB} : \text{AC} :: Q : P\,(= Q + R), \tag{1}$$

and therefore $\quad \text{AB} : \text{BC} :: Q : R; \qquad (2)$

at C introduce opposite forces each equal to Q and parallel to P; this will not alter the effect of the couple.

Now R at A and Q at C will balance $Q+R$ at B from (2) and (Art. 45); hence there remain the forces, Q, Q, acting on the arm, AC, which form a couple whose moment is equal to that of P, P, with arm, AB, since by (1) we have

$$P \times \mathrm{AB} = Q \times \mathrm{AC}.$$

Hence the effect of the couple is not altered.

Rem.—From the last three articles it appears that we may change a couple into another couple of equal moment, and transfer it to any position, either in its own plane or in a plane parallel to its own, without altering the effect of the couple. The couple must remain unchanged so far as concerns the *direction of rotation* which its forces would tend to give the arm, *i. e.*, the axis of the couple may be removed parallel to itself, to any position within the body acted on by the couple, while the direction of the axis from the plane of the couple is unaltered (Art. 50).

54. *A Force and a Couple acting in the same Plane on a Rigid Body are equivalent to a Single Force.*

Let the force be F and the couple (P, a), that is, P is the magnitude of each force in the couple whose arm is a. Then (Art. 53) the couple $(P, a) =$ the couple $\left(F, \frac{aP}{F}\right)$. Let this latter couple be moved till one of its forces acts in the same line as the given force, F, but in the opposite direction. The given force, F, will then be destroyed, and there will remain a force, F, acting in the same direction as the given one and at a perpendicular distance from it $= \frac{aP}{F}$.

GETTYSBURG COLLEGE
50696
Gettysburg, Pa.
LIBRARY

Cor.—*A force and a couple acting on a rigid body cannot produce equilibrium. A couple can be in equilibrium only with an equivalent couple. Equivalent couples are those whose moments are equal.**

The resultant of several couples is one which will produce the same effect singly as the component couples.

55. *To find the Resultant of any number of Couples acting on a Body, the Planes of the Couples being parallel to each other.*

Let P, Q, R, etc., be the forces, and a, b, c, etc., their arms respectively. Suppose all the couples transferred to the same plane (Art. 52); next, let them all be transferred so as to have their arms in the same straight line, and one extremity common (Art. 51); lastly, let them be replaced by other couples having the same arm (Art. 53). Let α be the common arm, and P_1, Q_1, R_1, etc., the new forces, so that

$$P_1\alpha = Pa, \quad Q_1\alpha = Qb, \quad R_1\alpha = Rc, \text{ etc.,}$$

then $$P_1 = P\frac{a}{\alpha}, \quad Q_1 = Q\frac{b}{\alpha}, \quad R_1 = R\frac{c}{\alpha}, \text{ etc.,}$$

i. e., the new forces are $P\frac{a}{\alpha}$, $Q\frac{b}{\alpha}$, $R\frac{c}{\alpha}$, etc., acting on the common arm α. Hence their resultant will be a couple of which each force equals

$$P\frac{a}{\alpha} + Q\frac{b}{\alpha} + R\frac{c}{\alpha} + \text{etc.,}$$

and the arm $= \alpha$, or the moment equals

$$Pa + Qb + Rc + \text{etc.}$$

If one of the couples, as Q, act in a direction opposite to the other couples its sign will be negative, and the force at each extremity of the arm of the resultant couple will be

---

* The moments of equivalent couples may have like or unlike signs

$$P\frac{a}{\alpha} - Q\frac{b}{\alpha} + R\frac{c}{\alpha} + \text{etc.}$$

Hence the moment of the resultant couple is equal to the algebraic sum of the moments of the component couples.

56. *To Find the Resultant of two Couples not acting in the same Plane.**

Let the planes of the couples be inclined to each other at an angle γ; let the couples be transferred in their planes so as to have the same arm lying along the line of intersection of the two planes; and let the forces of the couples thus transferred be P and Q. Let AB be the common arm. Let R be the resultant of the forces P and Q at A acting in the direction AR; and of P and Q at B acting in the direction BR. Then since P and Q at A are parallel to P and Q at B respectively, therefore R at A is parallel to R at B. Hence the two couples are equivalent to the single couple R, R, acting on the arm AB; and since $PAQ = \gamma$, we have

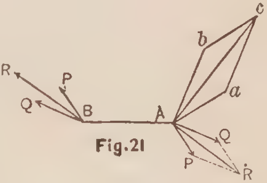

Fig.21

$$R^2 = P^2 + Q^2 + 2PQ\cos\gamma \text{ (Art. 30).} \qquad (1)$$

Draw Aa, Bb perpendicular to the planes of the couples P, P, and Q, Q, respectively, and proportional in length to their moments.

Draw Ac perpendicular to the plane of R, R, and in the same proportion to Aa, Bb, that the moment of the couple, R, R, is to those of P, P, and Q, Q. respectively. Then Aa, Ab, Ac, may be taken as the axes of P, P; Q, Q; and

---

* Todhunter's Statics, p. 42. Also Pratt's Mechanics, p. 25.

R, R, respectively (Art. 50). Now the three straight lines, Aa, Ac, Ab, make the same angles with each other that AP, AR, AQ make with each other; also they are in the same proportion in which

$$\mathrm{AB}\cdot P,\quad \mathrm{AB}\cdot R,\quad \mathrm{AB}\cdot Q \text{ are,}$$

or in which $$P,\ R,\ Q \text{ are.}$$

But R is the resultant of P and Q; therefore Ac is the diagonal of the parallelogram on Aa, Ab (Art. 30).

Hence if two straight lines, having a common extremity, represent the axes of two couples, that diagonal of the parallelogram described on these straight lines as adjacent sides which passes through their common extremity represents the axis of the resultant couple.

Cor.—Since R · AB is the axis or moment of the resultant couple, we have from (1)

$$R^2\cdot\overline{\mathrm{AB}}^2 = P^2\cdot\overline{\mathrm{AB}}^2 + Q^2\cdot\overline{\mathrm{AB}}^2 + 2P\cdot\mathrm{AB}\cdot Q\cdot\mathrm{AB}\cdot\cos\gamma. \quad (2)$$

If L and M represent the axes or moments of the component couples and G, that of the resultant couple, (2) becomes

$$G^2 = L^2 + M^2 + 2L\cdot M\cos\gamma. \quad (3)$$

Sch. 1.—If L, M, N, are the axes of three component couples which act in planes at right angles to one another, and G the axis of the resultant couple, it may easily be shown that

$$G^2 = L^2 + M^2 + N^2. \quad (4)$$

If λ, μ, ν be the angles which the axis of the resultant makes with those of the components, we have

$$\cos\lambda = \frac{L}{G},\quad \cos\mu = \frac{M}{G},\quad \cos\nu = \frac{N}{G}.$$

SCH. 2.—Hence, conversely any couple may be replaced by three couples acting in planes at right angles to one another; their moments being $G \cos \lambda$, $G \cos \mu$, $G \cos \nu$; where G is the moment of the given couple, and λ, μ, ν the angles its axis makes with the axes of the three couples.

Thus the composition and resolution of *couples* follow laws similar to those which apply to *forces*, the *axis* of the couple corresponding to the *direction* of the force, and the *moment* of the couple to the *magnitude* of the force.

57. Varignon's Theorem of Moments.—*The moment of the resultant of two component forces with respect to any point in their plane is equal to the algebraic sum of the moments of the two components with respect to the same point.*

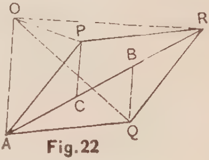

Fig. 22

Let AP and AQ represent two component forces; complete the parallelogram and draw the diagonal, AR, representing the resultant force. Let O be the origin of moments (Art. 46). Join OA, OP, OQ, OR, and draw PC and QB parallel to OA, and let $p =$ the perpendicular let fall from O to AR.

Now the moment of AP about O is the product of AP and the perpendicular let fall on it from O (Art. 46), which is double the area of the triangle, AOP (Art. 48). But the area of the triangle, AOP, $=$ the area of the triangle, AOC, since these triangles have the same base, AO, and are between the same parallels, AO and CP. Hence the moment of AP about $O =$ the moment of AC about $O = AC \cdot p$. Also the moment of AQ about O is double the area of the triangle, AOQ, $=$ double the area of the triangle, AOB, since the two triangles have the same base, AO, and are between the same parallels, AO and QB. Hence the moment of AQ about $O =$ the moment of AB

about $O = AB \cdot p$. Therefore the sum of the moments of AP and AQ about $O =$ the sum of the moments of AC and AB about $O = (AC + AB)p$, $= (AB + BR)p$, (since $AC = BR$ from the equal triangles APC and QBR) $= AR \cdot p =$ the moment of the resultant.

If the origin of moments fall *between* AP and AQ, the forces will tend to produce rotation in opposite directions, and hence their moments will have contrary signs (Art. 47). In this case the moment of the resultant = the *difference* of the moments of the components, as the student will find no difficulty in showing. Hence in either case the moment of the resultant is equal to the *algebraic* sum of the moments of the components.

Cor. 1.—If there are any number of component forces, we may compound them in order, taking any two of them first, then finding the resultant of these two and a third, and so on; and it follows that the sum of their moments (with their proper signs), is equal to the moment of the resultant.

Cor. 2.—If the origin of moments be on the line of action of the resultant, $p = 0$, and therefore the moment of the resultant $= 0$; hence the sum of the moments of the components is equal to zero. In this case the moments of the forces in one direction balance those in the opposite direction; *i. e.*, the forces that tend to produce rotation in one direction are counteracted by the forces that tend to produce rotation in the opposite direction, and there is no tendency to rotation.

Cor. 3.—If all the forces are in equilibrium the resultant $R = 0$, and therefore the moment of $R = 0$; hence the sum of the moments of the components is equal to zero, and there is no tendency to motion either of translation or rotation.

Cor. 4.—Therefore when the moment of the resultant $= 0$, we conclude either that the resultant $= 0$ (Cor. 3), or that it passes through the point taken as the origin of moments (Cor. 2).

58. Varignon's Theorem of Moments for Parallel Forces.—*The sum of the moments of two parallel forces about any point is equal to the moment of their resultant about the point.*

Let P and Q be two parallel forces acting at A and B, and R their resultant acting at G, and let O be the point about which moments are to be taken. Then (Art. 45) we have

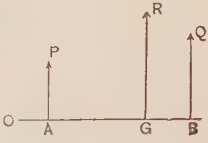

Fig.23

$$P \times \mathrm{AG} = Q \times \mathrm{BG},$$

$$\therefore \quad P(\mathrm{OG} - \mathrm{OA}) = Q(\mathrm{OB} - \mathrm{OG}),$$

$$\therefore \quad (P + Q)\,\mathrm{OG} = P \times \mathrm{OA} + Q \times \mathrm{OB},$$

$$\therefore \quad R \times \mathrm{OG} = P \times \mathrm{OA} + Q \times \mathrm{OB};$$

that is, the sum of the moments = the moment of the resultant.

Cor.—It follows that the algebraic sum of the moments of any number of parallel forces in one plane, with respect to a point in their plane, is equal to the moment of their resultant with respect to the point.

59. Centre of Parallel Forces.—*To find the magnitude, direction, and point of application of the resultant of any number of parallel forces acting on a rigid body in one plane.*

Let P_1, P_2, P_3, etc., denote the forces, M_1, M_2, M_3, etc., their points of application. Take any point in the plane of the forces as origin and draw the rectangular axes OX, OY. Let (x_1, y_1), (x_2, y_2), etc., be the points of application, M_1, M_2, etc. Join M_1M_2; and take the point M on M_1M_2, so that

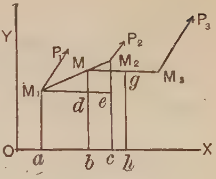

Fig.24

$$\frac{M_1M}{M_1M_2} = \frac{P_2}{P_1 + P_2}; \qquad (1)$$

then the resultant of P_1 and P_2 is $P_1 + P_2$, and it acts through M parallel to P_1 (Art. 45).

Draw M_1a, Mb, M_2c parallel, and M_1e perpendicular to the axis of y. Then we have

$$\frac{M_1M}{M_1M_2} = \frac{Md}{M_2e} = \frac{Mb - y_1}{y_2 - y_1}.$$

$$\therefore \quad Mb - y_1 = \frac{P_2}{P_1 + P_2}(y_2 - y_1);$$

$$\therefore \quad Mb = \frac{P_1y_1 + P_2y_2}{P_1 + P_2}; \qquad (2)$$

which gives the ordinate of the point of application of the resultant of P_1 and P_2.

Now since the resultant of P_1 and P_2, which is $P_1 + P_2$, acts at M, the resultant of $P_1 + P_2$ at M, and P_3 at M_3, is $P_1 + P_2 + P_3$ at g, and substituting in (2) $P_1 + P_2$, P_3, Mb, and y_3 for P_1, P_2, y_1, and y_2 respectively, we have

$$gh = \frac{(P_1 + P_2)\,Mb + P_3y_3}{P_1 + P_2 + P_3} = \frac{P_1y_1 + P_2y_2 + P_3y_3}{P_1 + P_2 + P_3}; \quad (3)$$

and this process may be extended to any number of parallel forces. Let R denote the resultant force and $\bar{y}$ the ordinate of the point of application; then we have

$$R = P_1 + P_2 + P_3 + \text{etc.} = \Sigma P.$$

$$\bar{y} = \frac{P_1 y_1 + P_2 y_2 + P_3 y_3 + \text{etc.}}{P_1 + P_2 + P_3 + \text{etc.}} = \frac{\Sigma Py}{\Sigma P}.$$

Similarly, if $\bar{x}$ be the abscissa of the point of application of the resultant, we have

$$\bar{x} = \frac{\Sigma Px}{\Sigma P}.$$

The values of $\bar{x}$, $\bar{y}$ are independent of the angles which the directions of the forces make with the axes. Hence if these directions be turned about the points of application of the forces, their parallelism being preserved, the point of application of the resultant will not move. For this reason the point $(\bar{x}, \bar{y})$ is called the *centre of parallel forces.* We shall hereafter have many applications in which its position is of great importance.

SCH. 1.—*The moment of a force with respect to a plane* is the product of the force into the perpendicular distance of its point of application from the plane. Thus, $P_1 y_1$ is the moment of the force P_1, in reference to the plane through OX perpendicular to OY. This must be carefully distinguished from the moment of a force with respect to a point. Hence the equations for determining the position of the *centre of parallel forces* show that *the sum of the moments of the parallel forces with respect to any plane, is equal to the moment of their resultant.*

SCH. 2.—*The moment of a force with respect to any line is the product* of the component of the force perpendicular to the line into the shortest distance between the line and the line of action of the force.

60. Conditions of Equilibrium of a Rigid Body acted on by Parallel Forces in one Plane.—Let P_1, P_2, P_3, etc., denote the forces. Take any point in the plane of the forces as origin, and draw rectangular axes, OX, OY, the latter parallel to the forces. Let A be the point where OX meets the direction of P_1, and let $OA = x_1$.

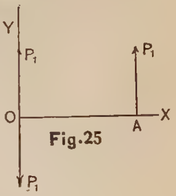

Fig. 25

Apply at O two opposing forces, each equal and parallel to P_1; this will not disturb the equilibrium. Then P_1 at A is replaced by P_1 at O along OY, and a couple whose moment is $P_1 \cdot OA$, *i. e.*, P_1x_1. The remaining forces, P_2, P_3, etc., may be treated in like manner. We thus obtain a set of forces, P_1, P_2, P_3, etc., acting at O along OY, and a set of couples, P_1x_1, P_2x_2, P_3x_3, etc., in the plane of the forces tending to turn the body from the axis of x to the axis of y. These forces are equivalent to a single resultant force $P_1 + P_2 + P_3 +$ etc., and the couples are equivalent to a single resultant couple, $P_1x_1 + P_2x_2 + P_3x_3 +$ etc. (Art. 55).

Hence denoting the resultant force by R, and the moment of the resultant couple by G, we have

$$R = P_1 + P_2 + P_3 + \text{etc.} = \Sigma P;$$

$$G = P_1x_1 + P_2x_2 + P_3x_3 + \text{etc.} = \Sigma Px;$$

that is, a system of parallel forces can be reduced to a single force and a couple, which (Art. 54, Cor.) cannot produce equilibrium. Hence, for equilibrium, the force and the couple must vanish; or

$$\Sigma P = 0, \quad \text{and} \quad \Sigma Px = 0.$$

Hence the conditions of equilibrium of a system of parallel forces acting on a rigid body in one plane are :

The sum of the forces must $= 0$.

The sum of the moments of the forces about every point in their plane must $= 0$.

61. Conditions of Equilibrium of a Rigid Body acted on by Forces in any direction in one Plane.— Let P_1, P_2, P_3, etc., be the forces acting at the points (x_1, y_1), (x_2, y_2), (x_3, y_3), etc., in the plane xy. Resolve the force P_1 into two components, X_1, Y_1, parallel to OX and OY respectively. Let the direction of Y_1 meet OX at M, and the direction of X_1 meet OY at N. Apply at O two opposing forces each equal and parallel to X_1, and also two opposing forces each equal and parallel to Y_1. Hence Y_1 at A_1, or M, is equivalent to Y_1 at O, and a couple whose moment is $Y_1 \cdot OM$; and X_1 at A_1, or N, is equivalent to X_1 at O, and a couple whose moment is $X_1 \cdot ON$.

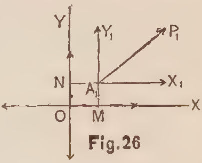

Fig.26

Hence Y_1 is replaced by Y_1 at O, and the couple Y_1x_1; and X_1 is replaced by X_1 at O, and the couple X_1y_1 (Art. 47). Therefore the force P_1 may be replaced by the components X_1, Y_1 acting at O, and the couple whose moment is

$$Y_1x_1 - X_1y_1,$$

and which equals the moment of P_1 about O (Art. 57).

By a similar resolution of all the forces we shall have them replaced by the forces (X_2, Y_2), (X_3, Y_3), etc., acting at O along the axes, and the couples

$$Y_2x_2 - X_2y_2, \quad Y_3x_3 - X_3y_3, \text{ etc.}$$

Adding together the couples or moments of P_1, P_2, etc., and denoting by G the moment of the resultant couple, we get the total moment

$$G = \Sigma\,(Yx - Xy).$$

If the sum of the components of the forces along OX is denoted by ΣX, and the sum of the components along OY by ΣY, the resultant of the forces acting at O is given by the equation

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2.$$

If a be the angle which R makes with the axis of X, we have

$$R \cos a = \Sigma X, \quad R \sin a = \Sigma Y;$$

$$\therefore \quad \tan a = \frac{\Sigma Y}{\Sigma X}.$$

Therefore, any system of forces acting in any direction in one plane on a rigid body may be reduced to a single force, R, and a single couple whose moment is G, which (Art. 54, Cor.) cannot produce equilibrium. Hence for equilibrium we must have $R = 0$, and $G = 0$, which requires that

$$\Sigma X = 0, \quad \Sigma Y = 0,$$

$$\Sigma\,(Yx - Xy) = 0.$$

Hence the conditions of equilibrium for a system of forces acting in any direction in one plane on a rigid body are :

The sum of the components of the forces parallel to each of two rectangular axes must $= 0$.

The sum of the moments of the forces round every point in their plane must $= 0$.

Cor.—Conversely, if the forces are in equilibrium the sum of the components of the forces parallel to any direction will $= 0$, and also the sum of the moments of the forces about any point will $= 0$.

62. Condition of Equilibrium of a Body under the Action of Three Forces in one Plane.—*If three forces maintain a body in equilibrium, their directions must meet in a point, or be parallel.*

Suppose the directions of two of the forces, P and Q, to meet at a point, and take moments round this point; then the moment of each of these two forces $= 0$; therefore the moment of the third force $R = 0$ (Art. 61, Cor.), which requires either that $R = 0$, or that it pass through the point of intersection of P and Q. If R is not $= 0$, it must pass through this point. Hence if any two of the forces meet, the third must pass through their point of intersection, and keep it at rest, and each force must be equal and opposite to the resultant of the other two. If the angles between them in pairs be p, q, r, the forces must satisfy the conditions

$$P : Q : R = \sin p : \sin q : \sin r \text{ (Art. 32).}$$

If two of the forces are parallel, the third must be parallel to them, and equal and directly opposed to their resultant.

EXAMPLES.

1. Suppose six parallel forces proportional to the numbers 1, 2, 3, 4, 5, 6 to act at points $(-2, -1)$, $(-1, 0)$, $(0, 1)$, $(1, 2)$, $(2, 3)$, $(3, 4)$; find the resultant, R, and the centre of parallel forces.

By Art. 59 we have

$$R = \Sigma P = 1 + 2 + \ldots 6 = 21;$$

$$\Sigma Px = -2 - 2 + 4 + 10 + 18 = 28;$$

$$\Sigma Py = -1 + 3 + 8 + 15 + 24 = 49.$$

$$\therefore \quad \bar{x} = \frac{\Sigma Px}{\Sigma P} = \frac{28}{21}; \quad \bar{y} = \frac{\Sigma Py}{\Sigma P} = \frac{49}{21}.$$

2. At the three vertices of a triangle parallel forces are applied which are proportional respectively to the opposite sides of the triangle; find the centre of these forces.

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be the vertices, and let a, b, c be the sides opposite to them; then

$$\bar{x} = \frac{ax_1 + bx_2 + cx_3}{a+b+c}; \quad \bar{y} = \frac{ay_1 + by_2 + cy_3}{a+b+c}.$$

3. If two parallel forces, P and Q, act in the same direction at A and B, (Fig. 14), and make an angle, θ, with AB, find the moment of each about the point of application of their resultant.

The moment of P with respect to G is

$$P \cdot AG \sin\theta \text{ (Art. 46).}$$

But from (1) of Art. 45, we have

$$\frac{P+Q}{Q} = \frac{AB}{AG};$$

$$\therefore \quad AG = \frac{Q}{P+Q} \cdot AB,$$

which in $P \cdot AG \sin\theta$ gives

$$\frac{PQ}{P+Q} \cdot AB \sin\theta,$$

for the moment of P which also equals the moment of Q

4. Two parallel forces, acting in the same direction, have their magnitudes 5 and 13, and their points of application, A and B, 6 feet apart. Find the magnitude of their resultant, and the point of application, G.

Ans. $R = 18,\ AG = 4\frac{1}{3},\ BG = 1\frac{2}{3}$.

5. On a straight rod, AF, there are suspended 5 weights of 5, 15, 7, 6, and 9 pounds respectively at the points A, B, D, E, F; $AB = 3$ feet, $BD = 6$ feet, $DE = 5$ feet, $EF = 4$ feet. Find the magnitude of the resultant, and the distance of its point of application, G, from A.

Ans. $R = 42$ pounds. $AG = 8\frac{3}{7}$ feet.

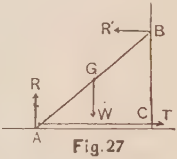

Fig. 27

6. A heavy uniform beam, AB, rests in a vertical plane, with one end, A, on a smooth horizontal plane and the other end, B, against a smooth vertical wall; the end, A, is prevented from sliding by a horizontal string of given length fastened to the end of the beam and to the wall; determine the tension of the string and the pressures against the horizontal plane and the wall.

Let $2a$ = the length of the beam, and let W be its weight, which as the beam is uniform, we may suppose to act at its middle point, G. Let R be the vertical pressure of the horizontal plane against the beam; and R' the horizontal pressure of the vertical wall, and T the tension of the horizontal string, AC; let BAC $= \alpha$, a known angle, since the lengths of the beam and the string are given. Then (Art. 61), we have

$$\text{for horizontal forces,}\quad T = R';$$

$$\text{for vertical forces,}\qquad W = R;$$

for moments about A (Art. 47), $2R'a \sin\alpha = Wa\cos\alpha$;

$$\therefore\quad R' = T = \frac{W}{2}\cot\alpha.$$

7. A heavy beam, AB $= a + b$, rests on two given smooth planes which are inclined at angles, α and β, to the horizon; required the angle θ which the beam makes with the horizontal plane, and the pressures on the planes.

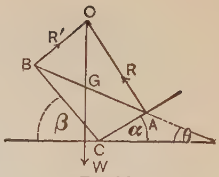

Fig. 28

Let a and b be the segments, AG and BG, of the beam, made by its centre of gravity, G; let R and R' be the pressures on the planes, AC and BC, the lines of action of which are perpendicular to the planes since they are smooth, and let W be the weight of the beam. Then we have

$$\text{for horizontal forces, } R \sin \alpha = R' \sin \beta; \qquad (1)$$

$$\text{for vertical forces, } R \cos \alpha + R' \cos \beta = W; \qquad (2)$$

$$\text{for moments about G, } Ra \cos(\alpha + \theta) = R'b \cos(\beta - \theta). \quad (3)$$

Dividing (3) by (1), we have

$$a \cot \alpha - a \tan \theta = b \cot \beta + b \tan \theta;$$

therefore,
$$\tan \theta = \frac{a \cot \alpha - b \cot \beta}{a + b},$$

and from (1) and (2) we have

$$R = \frac{W \sin \beta}{\sin(\alpha + \beta)}; \text{ and } R' = \frac{W \sin \alpha}{\sin(\alpha + \beta)}.$$

Otherwise thus: since the beam is in equilibrium under the action of only three forces, they must meet in a point O, (Art. 62), and therefore we obtain immediately from the geometry of the figure,

$$\frac{R}{W} = \frac{\sin \beta}{\sin(\alpha + \beta)}, \quad \therefore \; R = \frac{W \sin \beta}{\sin(\alpha + \beta)},$$

and $$\frac{R'}{W} = \frac{\sin \alpha}{\sin (\alpha + \beta)}, \qquad \therefore \quad R' = \frac{W \sin \alpha}{\sin (\alpha + \beta)}.$$

Also since the angles, GOA and GOB, are equal to α and β, respectively, and BGO $= \frac{\pi}{2} - \theta$, we have

$$(a + b) \cot \text{BGO} = a \cot \text{GOA} - b \cot \text{GOB};$$

therefore, $$\tan \theta = \frac{a \cot \alpha - b \cot \beta}{a + b}.$$

Hence, if $\frac{a}{b} = \frac{\tan \alpha}{\tan \beta}$, the beam will rest in a horizontal position.

8. A heavy uniform beam, AB, rests with one end, A, against a smooth vertical wall, and the other end, B, is fastened by a string, BC, of given length to a point, C, in the wall; the beam and the string are in a vertical plane; it is required to determine the pressure against the wall, the tension of the string, and the position of the beam and the string.

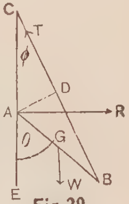

Fig. 29

Let $\quad \text{AG} = \text{GB} = a, \quad \text{AC} = x, \quad \text{BC} = b,$

weight of beam $= W$, tension of string $= T$, pressure of wall $= R$,

$$\text{BAE} = \theta, \quad \text{BCA} = \phi.$$

Then we have

$$\text{for horizontal forces,} \quad R = T \sin \phi; \tag{1}$$

$$\text{for vertical forces,} \quad W = T \cos \phi; \tag{2}$$

$$\text{for moments about A,} \quad Wa \sin \theta = T \cdot \text{AD} = Tx \sin \phi; \tag{3}$$

$$\therefore \quad a \sin \theta = x \tan \phi; \tag{4}$$

and by the geometry of the figure

$$\frac{b}{2a} = \frac{\sin \theta}{\sin \phi}; \tag{5}$$

$$\frac{x}{2a} = \frac{\sin (\theta - \phi)}{\sin \phi}. \tag{6}$$

Solving (4), (5), and (6), we get

$$x = \left[\frac{b^2 - 4a^2}{3}\right]^{\frac{1}{2}};$$

$$\cos \phi = \frac{2}{b}\left[\frac{b^2 - 4a^2}{3}\right]^{\frac{1}{2}};$$

$$\sin \theta = \frac{1}{2a}\left[\frac{16a^2 - b^2}{3}\right]^{\frac{1}{2}};$$

from which R and T become known. (Price's Anal. Mech's., Vol. I, p. 69).

To determine all the unknown quantities many problems in Statics require equations to be formed by *geometric* relations as well as *static* relations. Thus (1), (2), (3) are static equations, and (5) is a geometric equation.

9. A uniform heavy beam, AB $= 2a$, rests with one end, A, against the internal surface of a smooth hemispherical bowl, radius $= r$, while it is supported at some point in its length by the edge of the bowl; find the position of equilibrium.

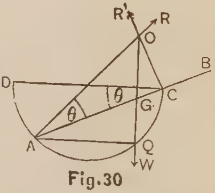

Fig. 30

The beam is kept in equilibrium by three forces, viz., the reaction, R, at A perpendicular to the *surface of contact*, (Art. 42) and therefore perpendicular to the bowl, the reaction, R', at C which, for the same reason, is perpendicular to the beam, and the weight W acting at G.

Let θ = the inclination of the beam to the horizon $= \angle ACD$. The solution will be most readily effected by resolving the forces along the beam and taking moments about C, by which we shall obtain equations free from the unknown reaction, R'. Then we have

$$\text{for forces along AB,}\quad R\cos\theta = W\sin\theta, \tag{1}$$

for moments about C,

$$R\cdot 2r\cos\theta\sin\theta = W(2r\cos\theta - a)\cos\theta. \tag{2}$$

From (1) we have

$$R = W\tan\theta,$$

which in (2) gives, after reducing,

$$2r\sin^2\theta - 2r\cos^2\theta + a\cos\theta = 0,$$

or,

$$4r\cos^2\theta - a\cos\theta - 2r = 0, \tag{3}$$

$$\therefore\quad \cos\theta = \frac{a \pm \sqrt{32r^2 + a^2}}{8r}.$$

Otherwise thus: since the beam is in equilibrium under the action of only three forces, they must meet in a point O (Art. 62). Draw the three forces AO, CO, GO, which keep the beam in equilibrium. Let the line, OG, meet the semicircle, DAC, in the point, Q. Then AQ is a horizontal line. Also

$$\angle QAG = \angle DCA = \theta,$$

therefore

$$\angle OAQ = 2\theta.$$

Hence

$$AQ = AO\cos 2\theta,$$

and also

$$AQ = AG\cos\theta;$$

therefore $$2r \cos 2\theta = a \cos \theta,$$

or $$4r \cos^2 \theta - a \cos \theta - 2r = 0,$$

which is the same as (3) obtained by the other method.

The student may prove that the reaction, R', at C $= W\frac{a}{2r}$.

10. Find the position of equilibrium of a uniform heavy beam, one end of which rests against a smooth vertical plane, and the other against the internal surface of a smooth spherical bowl.

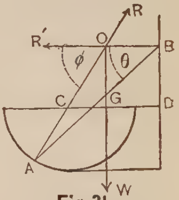

Fig. 31

The beam is in equilibrium under the action of three forces, the weight, W, acting at G, the reaction, R, at A, perpendicular to the surface and hence passing through the centre, C, and the reaction, R', of the vertical plane perpendicular to itself and hence horizontal.

Let the length of the beam, AB, $= 2a$, $r =$ the radius of the sphere, $d =$ CD, the distance of the centre of the sphere from the vertical wall, $W =$ the weight of the beam; and let $\theta =$ the required inclination of the beam to the horizon, and $\phi =$ the inclination of the radius AC to the horizon. Then we have

for vertical forces, $R \sin \phi = W$; (1)

for moments about B, $R \cdot 2a \sin (\phi - \theta) = W \cdot a \cos \theta$; (2)

Dividing (2) by (1) we have

$$\frac{2 \sin (\phi - \theta)}{\sin \phi} = \cos \theta,$$

or $$\tan \phi = 2 \tan \theta. \tag{3}$$

Then we have, from the geometry of the figure, the horizontal distance from A to the wall = the horizontal projection of AC + CD, that is,

$$2a \cos \theta = r \cos \phi + d. \tag{4}$$

From (3) and (4) a value of θ can be obtained, and hence the position of equilibrium.

Otherwise thus: since the beam is in equilibrium under the action of only three forces they must meet in a point, O. Geometry then gives us

$$2 \cot \mathrm{OGB} = \cot \mathrm{AOG} - \cot \mathrm{GOB} = \cot \mathrm{AOG},$$

or
$$2 \tan \theta = \tan \phi,$$

which is the same as (3).

63. Centre of Parallel Forces in Different Planes.

—*To find the magnitude, direction, and point of application of the resultant of any number of parallel forces acting on a rigid body.*

The theorem of Art. 59 is evidently true also in the case in which neither the parallel forces nor their fixed points of application lie in the same plane, hence, calling $\bar{z}$ the third co-ordinate of the point of application of the resultant, we have for the distance of the *centre of parallel forces* from the planes yz, zx, and xy,

$$\bar{x} = \frac{\Sigma Px}{\Sigma P}, \quad \bar{y} = \frac{\Sigma Py}{\Sigma P}, \quad \bar{z} = \frac{\Sigma Pz}{\Sigma P}.$$

Hence (Art. 59, Sch.) the equations for determining the position of the *centre of parallel forces* show that the *sum of the moments of the parallel forces with respect to any plane is equal to the moment of their resultant.*

64. Conditions of Equilibrium of a System of Parallel Forces Acting upon a Rigid Body in Space.—Let P_1, P_2, P_3, etc., denote the forces, and let them be referred to three rectangular axes, OX, OY, OZ; the last parallel to the forces; let (x_1, y_1, z_1), (x_2, y_2, z_2), etc., be the points of application of the forces, P_1, P_2, etc. Let the direction of P_1 meet the plane, xy, at M_1.

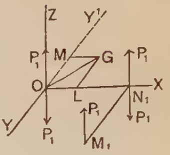

Fig.32

Draw M_1N_1 perpendicular to the axis of x meeting it at N_1. Apply at O, and also at N_1, two opposing forces each equal and parallel to P_1. Then the force P_1 at M_1 is replaced by

(1) P_1 at O along OZ;

(2) a couple formed of P_1 at M_1 and P_1 at N_1;

(3) a couple formed of P_1 at N_1 and P_1 at O.

The moment of the first couple is P_1y_1, and this couple may be transferred to the plane yz, which is parallel to its original plane, without altering its effect (Art. 52). The moment of the second couple is P_1x_1, and the couple is in the plane xz.

Replacing each force in this manner, the whole system will be equivalent to a force

$$P_1 + P_2 + P_3 + \text{etc., or } \Sigma P \text{ at } O \text{ along } OZ,$$

together with the couple

$$P_1y_1 + P_2y_2 + P_3y_3 + \text{etc., or } \Sigma Py, \text{ in the plane } yz,$$

and the couple

$$P_1x_1 + P_2x_2 + P_3x_3 + \text{etc., or } \Sigma Px \text{ in the plane } xz.$$

The first couple tends to turn the body from the axis of y to that of z round the axis of x, and the second couple tends to turn the body from the axis of x to that of z round the axis of y. It is customary to consider those couples as positive which tend to turn the body in the direction indicated by the natural order of the letters, *i. e.*, *positive* from x to y, round the z-axis; from y to z round the x-axis; and from z to x round the y-axis; and *negative* in the contrary direction.

Hence the moment of the first couple is $+\Sigma Py$, and therefore OX is its axis (Art. 50): and the moment of the second couple is $-\Sigma Px$, and therefore OY' is its axis. The resultant of these two couples is a single couple whose axis is found (Art. 56) by drawing OL (in the positive direction of the axis of x) $= \Sigma Py$, and OM (in the negative direction of the axis of y) $= \Sigma Px$, and completing the parallelogram $OLGM$. If OG, the diagonal, is denoted by G, we have

$$G = \sqrt{(\Sigma Px)^2 + (\Sigma Py)^2},$$

and $$R = \Sigma P;$$

R being the resultant force.

Now since this single force, R, and this single couple, G, cannot produce equilibrium (Art. 54, Cor.), we must have $R = 0$, and $G = 0$, and G cannot be $= 0$ unless $\Sigma Px = 0$ and $\Sigma Py = 0$; the conditions therefore of equilibrium are

$$R = 0,$$

$$\Sigma Px = 0, \quad \Sigma Py = 0.$$

Hence, the conditions of equilibrium of parallel forces in space are:

The sum of the forces must $= 0$.

The sum of the moments of the forces with respect to every plane parallel to them must $= 0$.

65. Conditions of Equilibrium of a System of Forces acting in any Direction on a Rigid Body in Space.—Let P_1, P_2, P_3, etc., denote the forces, and let them be referred to three rectangular axes, OX, OY, OZ; let (x_1, y_1, z_1), (x_2, y_2, z_2), etc., be the points of application of P_1, P_2, etc.

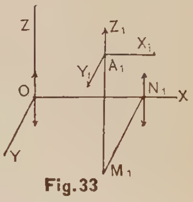

Fig. 33

Let A_1 be the point of application of P_1; resolve P_1 into components X_1, Y_1, Z_1, parallel to the co-ordinate axes. Let the direction of Z_1 meet the plane xy at M_1, and draw M_1N_1 perpendicular to OX. Apply at N_1 and also at O two opposing forces each equal and parallel to Z_1. Hence Z_1 at A_1 or M_1 is equivalent to Z_1 at O, and two couples of which the former has its moment $= Z_1 \times N_1M_1 = Z_1y_1$, and may be supposed to act in the plane yz, and the latter has its moment $= Z_1 \times ON_1 = - Z_1x_1$ and acts in the plane zx.

Hence Z_1 is replaced by Z, at O, a couple Z_1y_1 in the plane yz, and a couple $- Z_1x_1$ (Art. 64) in the plane zx. Similarly X_1 may be replaced by X_1 at O, a couple X_1z_1 in the plane zx, and a couple $- X_1y_1$ in the plane xy. And Y_1 may be replaced by Y_1 at O, a couple Y_1x_1 in the plane xy, and a couple $- Y_1z_1$ in the plane yz. Therefore the force P_1 may be replaced by X_1, Y_1, Z_1, acting at O, and three couples, of which the moments are, (Art. 56),

$Z_1y_1 - Y_1z_1$ in the plane yz, around the axis of x,
$X_1z_1 - Z_1x_1$ in the plane zx, around the axis of y,
$Y_1x_1 - X_1y_1$ in the plane xy, around the axis of z.

By a similar resolution of all the forces we shall have them replaced by the forces

$$\Sigma X, \quad \Sigma Y, \quad \Sigma Z,$$

acting at O along the axes, and the couples

$$\Sigma\,(Zy - Yz) = L, \text{ suppose, in the plane } yz,$$
$$\Sigma\,(Xz - Zx) = M, \text{ suppose, in the plane } zx,$$
$$\Sigma\,(Yx - Xy) = N, \text{ suppose, in the plane } xy.$$

Let R be the resultant of the forces which act at O; a, b, c, the angles its direction makes with the axes; then (Art. 38),

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2,$$

$$\cos a = \frac{\Sigma X}{R}, \quad \cos b = \frac{\Sigma Y}{R}, \quad \cos c = \frac{\Sigma Z}{R}.$$

Let G be the moment of the couple which is the resultant of the three couples, L, M, N; λ, μ, ν, the angles its axis makes with the co-ordinate axes; then (Art. 56, Sch.),

$$G^2 = L^2 + M^2 + N^2,$$

$$\cos\lambda = \frac{L}{G}, \quad \cos\mu = \frac{M}{G}, \quad \cos\nu = \frac{N}{G}.$$

Therefore any system of forces acting in any direction on a rigid body in space may always be reduced to a single force, R, and a single couple, G, and cannot therefore produce equilibrium (Art. 54, Cor.). Hence for equilibrium we must have $R = 0$ and $G = 0$; therefore

$$(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2 = 0,$$

and $$L^2 + M^2 + N^2 = 0.$$

These lead to the six conditions,

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

$$\Sigma\,(Zy - Yz) = 0, \quad \Sigma\,(Xz - Zx) = 0,$$

$$\Sigma\,(Yx - Xy) = 0.$$

EXAMPLES.

1. If the weights, 1, 2, 3, 4, 5 lbs., act perpendicularly to a straight line at the respective distances of 1, 2, 3, 4, 5 feet from one extremity, find the resultant, and the distance of its point of application from the first extremity.

Ans. $R = 15$ lbs., $x = 3\frac{2}{3}$ feet.

2. Four weights of 4, -7, 8, -3 lbs., act perpendicularly to a straight line at the points A, B, C, D, so that AB = 5 feet, BC = 4 feet, CD = 2 feet; find the resultant and its point of application, G.

Ans. $R = 2$ lbs., AG = 2 feet.

3. Two parallel forces of 23 and 42 lbs., act at the points A and B, 14 inches apart; find GB to three places of decimals. *Ans.* 4 954 ins.

4. Two weights of 3 cwts. 2 qrs. 15 lbs., and 1 cwt. 3 qrs. 25 lbs. are supported at the points A and B of a straight line, the length AB = 3 feet 7 inches; find AG to three places of decimals of feet. *Ans.* 1.268 ft.

5. A bar of iron 15 inches long, weighing 12 lbs., and of uniform thickness, has a weight of 10 lbs. suspended from one extremity; at what point must the bar be supported that it may just balance.

The weight of the bar acts at its centre.

Ans. $4\frac{1}{11}$ in. from the weight.

6. A bar of uniform thickness weighs 10 lbs., and is 5 feet long; weights of 9 lbs. and 5 lbs. are suspended from its extremities; on what point will it balance?

Ans. 5 in. from the centre of the bar.

7. A beam 30 feet long balances itself on a point at one-third of its length from the thicker end; but when a weight of 10 lbs. is suspended from the smaller end, the prop must

be moved two feet towards it, in order to maintain the equilibrium. Find the weight of the beam. *Ans.* 90 lbs.

8. A uniform bar, 4 feet long, weighs 10 lbs., and weights of 30 lbs. and 40 lbs. are appended to its two extremities; where must the fulcrum* be placed to produce equilibrium?
Ans. 3 in. from the centre of the bar.

9. A bar of iron, of uniform thickness, 10 ft. long, and weighing $1\frac{1}{2}$ cwt., is supported at its extremities in a horizontal position, and carries a weight of 4 cwt. suspended from a point distant 3 ft. from one extremity. Find the pressures on the points of support.
Ans. 3.55 cwt., and 1.95 cwt.

10. A bar, each foot in length of which weighs 7 lbs., rests upon a fulcrum distant 3 feet from one extremity; what must be its length, that a weight of $71\frac{1}{2}$ lbs. suspended from that extremity may just be balanced by 20 lbs. suspended from the other? *Ans.* 9 ft.

11. Five equal parallel forces act at 5 angles of a regular hexagon, whose diagonal is a; find the point of application of their resultant.

Ans. On the diagonal passing through the sixth angle, at a distance from it of $\frac{3}{5}a$.

12. A body, P, suspended from one end of a lever without weight, is balanced by a weight of 1 lb. at the other end of the lever; and when the fulcrum is removed through half the length of the lever it requires 10 lbs. to balance P; find the weight of P. *Ans.* 5 lbs. or 2 lbs.

13. A carriage wheel, whose weight is W and radius r, rests upon a level road; show that the force, F, necessary to draw the wheel over an obstacle, of height h, is

$$F = W\frac{\sqrt{2rh - h^2}}{r - h}.$$

---

* The support on which it rests.

14. A beam of uniform thickness, 5 feet long, weighing 10 lbs., is supported on two props at the ends of the beam; find where a weight of 30 lbs. must be placed, so that the pressures on the two props may be 15 lbs. and 25 lbs.

Ans. 10 ins. from the centre.

15. Forces of 3, 4, 5, 6 lbs. act at distances of 3 ins., 4 ins., 5 ins. 6 ins., from the end of a rod; at what distance from the same end does the resultant act?

Ans. $4\frac{7}{9}$ inches.

16. Four vertical forces of 4, 6, 7, 9 lbs. act at the four corners of a square; find the point of application of the resultant. *Ans.* $\frac{5}{13}$ of middle line from one of the sides.

17. A flat board 12 ins. square is suspended in a horizontal position by strings attached to its four corners, A, B, C, D, and a weight equal to the weight of the board is laid upon it at a point 3 ins. distant from the side AB and 4 ins. from AD; find the relative tensions in the four strings. *Ans.* As $\frac{3}{4} : \frac{1}{2} : \frac{1}{3} : \frac{5}{12}$.

18. A rod, AB, moves freely about the end, B, as on a hinge. Its weight, W, acts at its middle point, and it is kept horizontal by a string, AC, that makes an angle of 45° with it. Find the tension in the string. *Ans.* $\frac{W}{\sqrt{2}}$.

19. A rod 10 inches long can turn freely about one of its ends; a weight of 4 lbs. is slung to a point 3 ins. from this end, and the rod is held by a string attached to its free end and inclined to it at an angle of 120°; find the tension in the string when the rod is horizontal.

Ans. $\frac{4}{5}\sqrt{3}$ lbs.

20. Two forces of 3 lbs. and 4 lbs. act at the extremities of a straight lever 12 ins. long, and inclined to it at angles of 120° and 135° respectively; find the position of the fulcrum. *Ans.* $(8 - 3\sqrt{6}) \times 9.6$ ins. from one end.

21. Find the true weight of a body which is found to weigh 8 ozs. and 9 ozs. when placed in each of the scale-pans of a false balance. *Ans.* $6\sqrt{2}$ ozs.

22. A beam 3 ft. long, the weight of which is 10 lbs., and acts at its middle point, rests on a rail, with 4 lbs. hanging from one end and 13 lbs. from the other; find the point at which the beam is supported; and if the weights at the two ends change places, what weight must be added to the lighter to preserve equilibrium?

Ans. 12 ins. from one end; 27 lbs.

23. Two forces of 4 lbs. and 8 lbs. act at the ends of a bar 18 ins. long and make angles of 120° and 90° with it; find the point in the bar at which the resultant acts.

Ans. $\frac{72}{13}(4 - \sqrt{3})$ ins. from the 4 lbs. end.

24. A weight of 24 lbs. is suspended by two flexible strings, one of which is horizontal, and the other is inclined at an angle of 30° to the vertical. What is the tension in each string? *Ans.* $8\sqrt{3}$ lbs.; $16\sqrt{3}$ lbs.

25. A pole 12 ft. long, weighing 25 lbs., rests with one end against the foot of a wall, and from a point 2 ft. from the other end a cord runs horizontally to a point in the wall 8 ft. from the ground; find the tension of the cord and the pressure of the lower end of the pole.

Ans. 11.25 lbs.; 27.4 lbs.

26. A body weighing 6 lbs. is placed on a smooth plane which is inclined at 30° to the horizon; find the two directions in which a force equal to the body may act to produce equilibrium. Also find what is the pressure on the plane in each case.

Ans. A force at 60° with the plane, or vertically upwards; $R = 6\sqrt{3}$, or 0.

27. A rod, AB, 5 ft. long, without weight, is hung from a point, C, by two strings which are attached to its ends

and to the point; the string, AC, is 3 ft., and BC is 4 ft. in length, and a weight of 2 lbs. is hung from A, and a weight of 3 lbs. from B; find the tensions of the strings.

Ans. $\sqrt{5}$ lbs.; $2\sqrt{5}$ lbs.

28. Find the height of a cylinder, which can just rest on an inclined plane, the angle of which is 60°, the diameter of the cylinder being 6 ins. and its weight acting at the middle point of its axis. *Ans.* 3.46 ins.

29. Two equal weights, P, Q, are connected by a string which passes over two smooth pegs, A, B, situated in a horizontal line, and supports a weight, W, which hangs from a smooth ring through which the string passes; find the position of equilibrium.

Ans. The depth of the ring below the line

$$AB = \frac{W}{2\sqrt{4P^2 - W^2}} \cdot AB.$$

30. The resultant of two forces, P, Q, acting at an angle, θ, is $= (2m + 1)\sqrt{P^2 + Q^2}$; when they act at an angle, $\frac{\pi}{2} - \theta$, it is $= (2m - 1)\sqrt{P^2 + Q^2}$; show that $\tan\theta = \frac{m-1}{m+1}$.

31. A uniform heavy beam, AB $= 2a$, rests on a smooth peg, P, and against a smooth vertical wall, AD; the horizontal distance of the peg from the wall being h; find the inclination, θ, of the beam to the vertical, and the pressures, R and S, on the wall and peg.

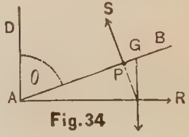

Fig.34

$$Ans.\ \theta = \sin^{-1}\left(\frac{h}{a}\right)^{\frac{1}{3}};\ S = W\left(\frac{a}{h}\right)^{\frac{1}{3}};\ R = W\frac{\sqrt{a^{\frac{2}{3}} - h^{\frac{2}{3}}}}{h^{\frac{1}{3}}}.$$

32. Two equal smooth cylinders rest in contact on two smooth planes inclined at angles, α and β, to the horizon;

find the inclination, θ, to the horizon of the line joining their centres. *Ans.* $\tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta)$.

33. A beam, 5 ft. long, weighing 5 lbs., rests on a vertical prop, CD = $2\frac{1}{2}$ ft.; the lower end, A, is on a horizontal plane, and is prevented from sliding by a string. AD = $3\frac{1}{3}$ ft.; find the tension of the string.

Ans. $T = \frac{36}{25}$ lbs.

34. A uniform beam, AB, is placed with one end, A, inside a smooth hemispherical bowl, with a point, P, resting on the edge of the bowl. If AB = 3 times the radius R, find AP. *Ans.* AP = 1.838 R.

35. A body, weight W, is suspended by a cord, length l, from the point A, in a horizontal plane, and is thrust out of its vertical position by a rod without weight, acting at another point, B, in the horizontal plane, such that AB = d, and making the angle, θ, with the plane; find the tension, T, of the cord. *Ans.* $T = W\frac{l}{d}\cot\theta$.

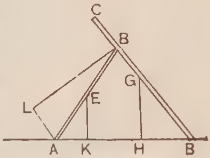

Fig. 35

36. Two heavy uniform bars, AB and CD, movable in a vertical plane about their extremities, A, D, which rest on a horizontal plane and are prevented from sliding on it; find their position of equilibrium when leaning against each other.

Let the bars rest against each other at B, and let AD = a, AB = b, CD = c, BD = x, W and W_1 = the weights of AB and CD, respectively acting at their middle points; then we have

$$2x^3 W(a^2 + b^2 - x^2) = cW_1(a^2 + x^2 - b^2)(b^2 + x^2 - a^2),$$

which is an equation of the fifth degree, and hence always has one real root, the value of which may be determined when numbers are put for a, b, and c.

37. A parabolic curve is placed in a vertical plane with its axis vertical and vertex downwards, and inside of it, and against a peg in the focus, and against the concave arc, a smooth uniform and heavy beam rests; required the position of equilibrium.

Let PB be the beam, of length l, and of weight W, resting on the peg at the focus, F; let AF $= p$ and AFP $= \theta$.

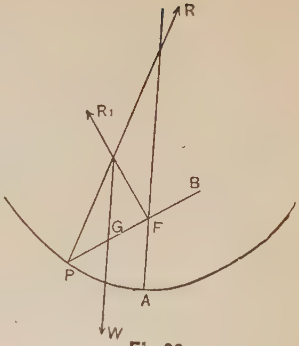

Fig. 36

Ans. $\theta = 2\cos^{-1}\left(\frac{p}{l}\right)^{\frac{1}{4}}$.

38. Find the form of the curve in a vertical plane such that a heavy bar resting on its concave side and on a peg at a given point, say the origin, may be at rest in all positions.

Ans. $r = \frac{1}{2}l + k\sec\theta$, in which $l =$ the length of the bar, k an arbitrary constant, and θ the inclination of the bar to the vertical. It is the equation of the conchoid of Nicomedes.

39. A rod whose centre of gravity is not its middle point is hung from a smooth peg by means of a string attached to its extremities; find the position of equilibrium.

Ans. There are two positions in which the rod hangs vertically, and there is a third thus defined:—Let F be the extremity of the rod remote from the centre of gravity, k the distance of the centre of gravity from the middle point of the rod, $2a$ the length of the string, and $2c$ the length of the rod; then measure on the string a length FP from F equal to $a\left(1 + \frac{k}{c}\right)$, and place the point P over the peg. This will define a third position of equilibrium.

40. A smooth hemisphere is fixed on a horizontal plane, with its convex side turned upwards and its base lying in the plane. A heavy uniform beam, AB, rests against the hemisphere, its extremity A being just out of contact with the horizontal plane. Supposing that A is attached to a rope which, passing over a smooth pulley placed vertically over the centre of the hemisphere, sustains a weight, find the position of equilibrium of the beam, and the requisite magnitude of the suspended weight.

Ans. Let W be the weight of the beam, $2a$ its length, P the suspended weight, r the radius of the hemisphere, h the height of the pulley above the plane, θ and ϕ the inclinations of the beam and rope to the horizon; then the position of equilibrium is defined by the equations.

$$r \operatorname{cosec} \theta = h \cot \phi, \tag{1}$$

$$r \operatorname{cosec}^2 \theta = a (\tan \phi + \cot \theta), \tag{2}$$

which give the single equation for θ,

$$r (r - a \sin \theta \cos \theta) = ah \sin^3 \theta. \tag{3}$$

Also
$$P = W \frac{\sin \theta}{\cos (\phi - \theta)}$$

$$= W \frac{a \sin^2 \theta \sqrt{r^2 + h^2 \sin^2 \theta}}{r^2}. \tag{4}$$

41. If, in the last example, the position and magnitude of the beam be given, find the locus of the pulley.

Ans. A right line joining A to the point of intersection of the reaction of the hemisphere and W.

42. If, in the same example, the extremity, A, of the beam rest against the plane, state how the nature of the problem is modified, and find the position of equilibrium.

Ans. The suspended weight must be given, instead of being a result of calculation. Equation (1) still holds, but not (2); and the position of equilibrium is defined by the equation

$$Ph^2 \cos^3 \phi = War \sin^2 \phi.$$

43. If the fixed hemisphere be replaced by a fixed sphere or cylinder resting on the plane, and the extremity of the beam rest on the ground, find the position of equilibrium.

Ans. If h denote the vertical height of the pulley above the point of contact of the sphere or cylinder with the plane, we have

$$r \cot \frac{\theta}{2} = h \cot \phi,$$

$$Pr \left(1 + \cot \frac{\theta}{2} \cot \theta\right) \cos \phi = Wa \cos \theta.$$

44. One end, A, of a heavy uniform beam rests against a smooth horizontal plane, and the other end, B, rests against a smooth inclined plane; a rope attached to B passes over a smooth pulley situated in the inclined plane, and sustains a given weight; find the position of equilibrium.

Let θ be the inclination of the beam to the horizon, α the inclination of the inclined plane, W the weight of the beam, and P the suspended weight; then the position of equilibrium is defined by the equation

$$\cos \theta \, (W \sin \alpha - 2P) = 0. \qquad (1)$$

Hence we draw two conclusions:—

(a) If the given quantities satisfy the equation $W \sin \alpha - 2P = 0$, the beam will rest in all positions.

(b) There is one position of equilibrium, namely, that in which the beam is vertical.

This position requires that both planes be conceived as prolonged through their line of intersection.

45. A uniform beam, AB, movable in a vertical plane about a smooth horizontal axis fixed at one extremity, A, is

attached by means of a rope BC, whose weight is negligible, to a fixed point C in the horizontal line through A, such that AB = AC; find the pressure on the axis.

Ans. If $\theta = \angle CAB$, W = weight of beam, the reaction is

$$\tfrac{1}{2} W \sqrt{4 \sin^2 \frac{\theta}{2} + \sec^2 \frac{\theta}{2}}.$$

CHAPTER IV.

CENTRE OF GRAVITY* (CENTRE OF MASS).

66. Centre of Gravity.—Gravity is the name given to the force of attraction which the earth exerts on all bodies; the effects of this force are twofold, (1) statical in virtue of which all bodies exert pressure, and (2) kinetical in virtue of which bodies if unsupported, will fall to the ground (Art. 15). The force of gravity varies slightly from place to place on the earth's surface (Art. 23); but at each place it is a force exerted upon every body and upon every particle of the body in directions that are normal to the earth's surface, and which therefore converge towards the earth's centre; but as this centre is very distant compared with the distance between the particles of any body of ordinary magnitude, the convergence is so small that the lines in which the force of gravity acts are sensibly parallel.

The centre of gravity of a body is the point of application of the resultant of all the forces of gravity which act upon every particle of the body; and since these forces are practically parallel, the problem of finding its position may be treated in the same way as that of finding the centre of a system of parallel forces (Arts. 45, 59, 63). The centre of gravity may also be defined as *the point at which the whole weight of a body acts.* If the body be supported at this point it will rest in any position whatever.

The weight of a body is the resultant of all the forces of gravity which act upon every particle of it, and is equal in magnitude and directly opposite to the force which will just support the body. Since the centre of gravity is here regarded as the centre of parallel forces, it is more truly conceived of as the "centre of mass;" yet in deference to usage we shall call the point the "centre of gravity."

---

* Called also *Centre of Mass* and *Centre of Inertia*; and the term *Centroid* has lately come into use to designate it.

67. Planes of Symmetry.—Axes of Symmetry.

—If a homogeneous body be *symmetrical* with reference to any plane, the centre of gravity is in that plane.

If two or more such planes of symmetry intersect in one line, or *axis of symmetry,* the centre of gravity is in that axis.

If three or more planes of symmetry intersect each other in a *point,* that point is the centre of gravity.

By observing these principles of the symmetry of the figure there are many cases where the centre of gravity is known at once; thus, the centre of gravity of a straight line is its middle point. The centre of gravity of a circle or of its circumference, or of a sphere or of its surface, is its centre. The centre of gravity of a parallelogram or of its perimeter is the point in which the diagonals intersect. The centre of gravity of a cylinder or of its surface is the middle of its axis; and in a similar manner we shall frequently conclude from the symmetry of the figure, that the centre of gravity of a body is in a particular line which can be at once determined.

When we speak of the centre of gravity of a line, we are really considering a *material* line of the same density and thickness throughout, whose section is infinitesimal; and when we consider the centre of gravity of any surface, we are really considering the surface as a thin uniform lamina, the thickness of which, being uniform, can be neglected.

68. Body Suspended from a Point.

—*When a body is suspended from a point about which it can turn freely, it will rest with its centre of gravity in the vertical line passing through the point of suspension.* For, if the point of suspension and the centre of gravity are not in a vertical line, the weight acting vertically downwards at the centre of gravity and the reaction of the point of suspension vertically upwards form a statical couple and hence there will be rotation.

69. Body Supported on a Surface.—*When a body is placed on a surface it will stand or fall according as the vertical line through the centre of gravity falls within or without the base.* For if it falls within the base the reaction of the surface upward and the action of the weight downward will be in the same vertical line, and so there will be equilibrium. But if it falls without the base the reaction of the surface upward and the action of the weight downward form a statical couple and hence the body will rotate and fall.

70. Different Kinds of Equilibrium.—According to the proposition just proved (Art. 69) a body ought to rest upon a single point without falling, provided that its centre of gravity is placed in the vertical line through the point which forms its base. And, in fact, a body so situated would be, mathematically speaking, in a position of equilibrium, though practically the equilibrium would not subsist. The body would be moved from its position by the least force, and if left to itself it would depart further from it, and never return to that position again. This kind of equilibrium, and that which is practically possible, are distinguished by the names of *unstable* and *stable*. Thus an egg on either end is in a position of *unstable* equilibrium, but when resting on its side it is in a position of *stable* equilibrium. The distinction may be defined generally as follows:

When the body is in such a position that if *slightly* displaced it tends to *return* to its original position, the equilibrium is *stable*. When it tends to move further away from its original position, its equilibrium is *unstable.* When it *remains in its new position,* its equilibrium is *neutral.* A sphere or cylindrical roller, resting on a horizontal surface, is in neutral equilibrium. In *stable equilibrium the centre of gravity occupies the lowest possible position; and in unstable it occupies the highest position.*

We shall first give a few elementary examples.

71. Given the Centres of Gravity of two Masses, M_1 and M_2, to find the Centre of Gravity of the two Masses as one System.—Let g_1, denote the centre of gravity of the mass M_1. and g_2 the centre of gravity of the mass M_2. Join $g_1\,g_2$ and divide it at the point, G, so that $\frac{Gg_1}{Gg_2} = \frac{M_2}{M_1}$, then G is the centre of gravity of the two masses as one system (Art. 45).

72. Given the Centre of Gravity of a Body of Mass, M, and also the Centre of Gravity of a part of the Body of Mass, m, to find the Centre of Gravity of the remainder.—Let G denote the centre of gravity of the mass, M, and g_1 the centre of gravity of the mass, m_1. Join Gg_1 and produce it through G to g_2, so that $\frac{Gg_2}{Gg_1} = \frac{m_1}{M - m_1}$, then g_2 is the centre of gravity of the remainder (Art. 45).

73. Centre of Gravity of a Triangular Figure of Uniform Thickness and Density.—Let ABC be the triangle; bisect BC in D, and join AD; draw any line *bdc* parallel to BC; then it is evident that this line will be bisected by AD in *d*, and will therefore have its centre of gravity at *d*; similarly every line in the triangle parallel to BC will have its centre of gravity in AD, and therefore the centre of gravity of the triangle must be somewhere in AD.

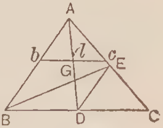

Fig. 37

In like manner the centre of gravity must lie on the line BE which joins B to the middle point of AC. It is therefore at the intersection, G, of AD and BE.

Join DE, which will be parallel to AB; then the triangles, ABG, DEG, are similar; therefore

$$\frac{AG}{GD} = \frac{AB}{DE} = \frac{BC}{DC} = \frac{2}{1};$$

or

$$GD = \tfrac{1}{2}AG = \tfrac{1}{3}AD.$$

Hence, *to find the centre of gravity of a triangle, bisect any side, join the point of bisection with the opposite angle, the centre of gravity lies one third the way up this bisection.*

Cor. 1.—If three equal particles be placed at the vertices of the triangle ABC their centre of gravity will coincide with that of the triangle.

For, the centre of gravity of the two equal particles at B and C is the middle point of BC, and the centre of gravity of the three lies on the line joining this point to A. Similarly, it lies on the line joining B to the middle of AC. Therefore, etc.

Cor. 2.—The centre of gravity of any plane polygon may be found by dividing it into triangles, finding the centre of gravity of each triangle, and then by Art. 59 deducing the centre of gravity of the whole figure.

Cor. 3.—Let the co-ordinates of A, referred to any axes, be x_1, y_1, z_1; those of B, x_2, y_2, z_2; and those of C, x_3, y_3, z_3; then (Art. 59), the co-ordinates, $\bar{x}$, $\bar{y}$, $\bar{z}$, of the centre of gravity of three equal particles placed at A, B, C, respectively, are

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3}; \quad \bar{y} = \frac{y_1 + y_2 + y_3}{3};$$

$$\bar{z} = \frac{z_1 + z_2 + z_3}{3};$$

which are also the co-ordinates of the centre of gravity of the triangle ABC (Cor. 1).

74. Centre of Gravity of a Triangular Pyramid of Uniform Density.—Let D–ABC be a triangular pyramid; bisect AC at E; join BE, DE; take EF $= \frac{1}{3}$EB, then F is the centre of gravity of ABC (Art. 73). Join FD; draw ab, bc, ca parallel to AB, BC, CA respectively, and let DF meet the plane, abc, at f; join bf and produce it to meet DE at e. Then since in the triangle ADC, ac is parallel to AC, and DE bisects AC, e is the middle point of ac; also

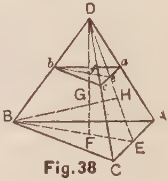

Fig. 38

$$\frac{bf}{\mathrm{BF}} = \frac{\mathrm{D}f}{\mathrm{DF}} = \frac{ef}{\mathrm{EF}};$$

but $\mathrm{EF} = \frac{1}{2}\mathrm{BF}$,

therefore $ef = \frac{1}{2}bf$;

therefore f is the centre of gravity of the triangle abc (Art. 73). Now if we suppose the pyramid to be divided by planes parallel to ABC into an indefinitely great number of triangular laminæ, each of these laminæ has its centre of gravity in DF. Hence the centre of gravity of the pyramid is in DF.

Again, take EH $= \frac{1}{3}$ED; join HB cutting DF at G. Then, as before the centre of the pyramid must be on BH. It is therefore at the intersection, G, of the lines DF and BH.

Join FH; then FH is parallel to DB. Also, EF $= \frac{1}{3}$EB, therefore FH $= \frac{1}{3}$DB; and in the similar triangles, FGH and BGD, we have

$$\frac{\mathrm{FG}}{\mathrm{DG}} = \frac{\mathrm{FH}}{\mathrm{DB}} = \frac{1}{3};$$

therefore $\mathrm{FG} = \frac{1}{3}\mathrm{DG} = \frac{1}{4}\mathrm{DF}$.

Hence, *the centre of gravity of the pyramid is one-fourth of the way up the line joining the centre of gravity of the base with the vertex.* (Todhunter's Statics, p. 108. Also Pratt's Mechanics, p. 53.)

Cor. 1.—The centre of gravity of four equal particles placed at the vertices of the pyramid coincides with the centre of gravity of the pyramid.

Cor. 2.—Let (x_1, y_1, z_1) be one of the vertices; (x_2, y_2, z_2) a second vertex, and so on; let $(\bar{x}, \bar{y}, \bar{z})$ be the centre of gravity of the pyramid; then (Art. 59)

$$\bar{x} = \tfrac{1}{4}(x_1 + x_2 + x_3 + x_4),$$

$$y = \tfrac{1}{4}(y_1 + y_2 + y_3 + y_4),$$

$$\bar{z} = \tfrac{1}{4}(z_1 + z_2 + z_3 + z_4).$$

Cor. 3.—The perpendicular distance of the centre of gravity of a triangular pyramid from the base is equal to $\frac{1}{4}$ of the height of the pyramid.

75. Centre of Gravity of a Cone of Uniform Density having any Plane Base.—Consider a pyramid whose base is a polygon of any number of sides. Divide the base into triangles; join the vertex of the pyramid with the vertices of all the triangles; then we may consider the pyramid as composed of a number of triangular pyramids. Now the centre of gravity of each of these triangular pyramids lies in a plane whose distance from the base is one-fourth of the height of the pyramid (Art. 74, Cor. 3); therefore the centre of gravity of the whole pyramid lies in this plane, *i. e.*, its perpendicular distance from the base is one-fourth of the height of the pyramid.

Again, if we suppose the pyramid to be divided into an indefinitely great number of laminæ, as in Art. 74, each of these laminæ has its centre of gravity on the right line joining the vertex to the centre of gravity of the base; and hence the centre of gravity of the whole pyramid lies on this line, and hence it must be one-fourth the way up this line. There is no limit to the number of sides of the polygon which forms the base of the pyramid, and hence they may form a continuous curve.

Therefore, *the centre of gravity of a cone whose base is any plane curve whatever is found by joining the centre of gravity of the base to the vertex, and taking a point one-fourth of the way up this line.*

76. Centre of Gravity of the Frustum of a Pyramid.—Let ABC–*abc* (Fig. 38) be the frustum, formed by the removal of the pyramid, D–*abc*, from the whole pyramid, D–ABC; let h_1 and H be the perpendicular heights of these pyramids, respectively; let m and M denote their masses; and let z_1, z_2, $\bar{z}$ denote the perpendicular distances of the centres of gravity of the pyramids D–ABC, and D–*abc*, and the frustum, from the base; then we have (Art. 59, Sch. 1)

$$Mz_1 = \bar{z}(M - m) + mz_2;$$

or

$$\bar{z} = \frac{Mz_1 - mz_2}{M - m}. \tag{1}$$

But

$$z_1 = \frac{\mathrm{H}}{4};$$

$$z_2 = (H - h_1) + \frac{h_1}{4} = H - \tfrac{3}{4}h_1.$$

Also, the masses of the pyramids are to each other as their volumes* by (1) of Art. 10, and therefore as the cubes of their heights. Hence (1) becomes

---

* If the bodies are homogeneous, the volumes or the weights are proportional to the masses, and may be substituted for them.

$$z = \frac{\frac{1}{4}H^4 - (H - \frac{3}{4}h_1)\,h_1^3}{H^3 - h_1^3}$$

$$= \frac{1}{4} \cdot \frac{H^4 - 4Hh_1^3 + 3h_1^4}{H^3 - h_1^3}$$

$$= \frac{H - h_1}{4} \cdot \frac{H^2 + 2Hh_1 + 3h_1^2}{H^2 + Hh_1 + h_2}. \qquad (2)$$

Instead of the heights we may use any two corresponding lines in the lower and upper bases, to which the heights are proportional, as for example AB and *ab*. Denoting these lines by a and b, and the altitude of the frustum by h, (2) becomes

$$z = \frac{h}{4} \cdot \frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}. \qquad (3)$$

This is true of a frustum of a pyramid on any base, a and b being homologous sides of the two ends, and hence it is true of the frustum of a cone standing on any plane base.

EXAMPLES.

1. Find the centre of gravity of a trapezoid in terms of the lengths of the two parallel sides, a and b, and of the line, h, joining their middle points.

Take moments with reference to the longer parallel side.

Ans. On the line bisecting the parallel sides and at a distance from its lower end $= \dfrac{h}{3} \cdot \dfrac{a + 2b}{a + b}$.

2. If out of any cone a similar cone is cut so that their axes are in the same line and their bases in the same plane, find the height of the centre of gravity of the remainder above the base.

Take moments with reference to the base.

Ans. $\frac{1}{4} \cdot \frac{h^4 - h'^4}{h^3 - h'^3}$, where h, is the height of the original cone, and h', the height of that which is cut out of it.

3. If out of any cone another cone is cut having the same base and their axes in the same line, find the height of the centre of gravity of the remainder above the base.

Ans. $\frac{1}{4}(h + h_1)$, where h and h_1 are the respective heights of the original cone and the one that is cut out of it.

4. If out of any right cylinder a cone is cut of the same base and height, find the centre of gravity of the remainder.

Ans. $\frac{3}{8}$ths of the height above the base.

77. Investigations Involving Integration. — The general formulæ for the co-ordinates of the centre of gravity vary according as we consider a material line, an area or thin lamina, or a solid; and assume different forms according to the manner in which the matter is supposed to be divided into infinitesimal elements.

In either case the principle is the same; the quantity of matter is divided into an infinite number of infinitesimal elements, the mass of the element being dm; multiplying the element by its co-ordinate, x, for example, we get $x \cdot dm$, which is the moment of the element* with respect to the plane yz (Art. 63); and $\int x \cdot dm$ is the sum of the moments of all the elements with respect to the plane yz, and which corresponds to ΣPx of Art. 63. Also, $\int dm$ is the sum of the masses of all the elements which correspond to ΣP of the same Article. Hence, dividing the former by the latter we have

---

* The moment of the force acting on element dm is strictly $dm \cdot g \cdot x$, but since the constant g appears in both terms of expression for co-ordinates of centre of gravity, it may be omitted and it becomes more convenient to speak of the *moment of the element*, meaning by it the product of the mass of the element dm. and its arm, x. The moment of an element measures its effect in determining the position of the centre of gravity.

$$\bar{x} = \frac{\int x \cdot dm}{\int dm}. \qquad (1)$$

Similarly

$$\bar{y} = \frac{\int y \cdot dm}{\int dm}, \qquad (2)$$

$$\bar{z} = \frac{\int z \cdot dm}{\int dm}; \qquad (3)$$

the limits of integration being determined by the form of the body; the sign, $\int$, is used as a general symbol of summation, to be replaced by the symbols of single, double, or triple integration, according as dm denotes the mass of an elementary length or surface or solid. Hence, *the co-ordinate of the centre of gravity referred to any plane is equal to the sum of the moments of the elements of the mass referred to the same plane divided by the sum of the elements, or the whole mass.* If the body has a plane of symmetry (Art. 67), we may take it to be the plane xy, and only (1) and (2) are necessary. If it has an axis of symmetry we may take it to be the axis of x, and only (1) is necessary.

78. Centre of Gravity of the Arc of a Curve.—If the body whose centre of gravity we want is a material line in the form of the arc of any curve, dm denotes the mass of an elementary length of the curve.

Let ds = the length of an element of the curve; let k = the area of a normal section of the curve at the point (x, y, z), and let ρ = the density of the matter at this point. Then (Art. 11), we have $dm = k\rho ds$, which is the mass of the element; multiplying this mass by its co-ordinate, x, for example, we have the moment of the element, $(k\rho x ds)$, with respect to the plane, yz.

Hence, substituting for dm in (1), (2), (3), of Art. 77, the linear element, $k\rho ds$, we obtain, for the position of the centre of gravity of a body in the form of any curve, the equations

$$\bar{x} = \frac{\int k\rho x ds}{\int k\rho ds}, \tag{1}$$

$$\bar{y} = \frac{\int k\rho y ds}{\int k\rho ds}, \tag{2}$$

$$z = \frac{\int k\rho z ds}{\int k\rho ds}. \tag{3}$$

The quantities k and ρ must be given as functions of the position of the point (x, y, z) before the integrations can be performed.

If the curve is of double curvature all three equations are required. If it is a plane curve, we may take it to be in the plane xy, and (1) and (2) are sufficient to determine the centre of gravity, since $\bar{z} = 0$. If the curve has an axis of symmetry, the axis of x may be made to coincide with it, and (1) is sufficient.

EXAMPLES.

1. To find the centre of gravity of a circular arc of uniform thickness and density.

Let BC be the arc, A its middle point, and O the centre of the circle. Then as the arc is symmetrical with respect to OA its centre of gravity must lie on this line. Take the origin at O, and OA as axis of x. Then, since k and ρ are constant, (1) becomes

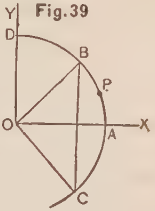
Fig.39

$$\bar{x} = \frac{\int x ds}{\int ds}; \tag{1}$$

x being the co-ordinate of any point, P, in the arc. Let θ be the angle POA, and a the radius of the circle, and let a = the angle BOA. Then

$$x = a\cos\theta,$$

and $$ds = a\,d\theta.$$

Hence $$\bar{x} = \frac{\int_{-\alpha}^{\alpha} a^2\cos\theta\,d\theta}{\int_{-\alpha}^{\alpha} a\,d\theta} = a\,\frac{\int_{-\alpha}^{\alpha}\cos\theta\,d\theta}{\int_{-\alpha}^{\alpha} d\theta} = a\,\frac{\sin\alpha}{\alpha}.$$

Therefore, *the distance of the centre of gravity of the arc of a circle from the centre is the product of the radius and the chord of the arc divided by the length of the arc.*

COR.—The distance of the centre of gravity of a semicircle from the centre is $\frac{2a}{\pi}$.

2. Find the centre of gravity of the quadrant, AD, (Fig. 39), referred to the co-ordinate axes OX, OY.

The equation of the circle is

$$x^2 + y^2 = a^2.$$

$$\therefore\quad \frac{dx}{y} = \frac{dy}{-x} = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{x^2 + y^2}} = \frac{ds}{a};$$

$$\therefore\quad x\,ds = \frac{ax\,dx}{y},$$

$$y\,ds = a\,dx,$$

and $$ds = \frac{a\,dx}{y};$$

which in (1) and (2), after canceling k and ρ, give

$$\bar{x} = \frac{\int_0^a \frac{x\,dx}{\sqrt{a^2 - x^2}}}{\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}} = \frac{\left[-(a^2 - x^2)^{\frac{1}{2}}\right]_0^a}{\left[\sin^{-1}\frac{x}{a}\right]_0^a} = \frac{2a}{\pi},$$

$$\bar{y} = \frac{\int_0^a dx}{\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}} = \frac{\left[x\right]_0^a}{\left[\sin^{-1}\frac{x}{a}\right]_0^a} = \frac{2a}{\pi}.$$

3. Find the centre of gravity of the arc of a cycloid.

Take the origin at the starting point of the cycloid, and let the base be taken as the axis of x. The equation of the curve is

$$x = a \operatorname{vers}^{-1}\frac{y}{a} - (2ay - y^2)^{\frac{1}{2}};$$

$$\therefore \quad \frac{dx}{y^{\frac{1}{2}}} = \frac{dy}{(2a - y)^{\frac{1}{2}}} = \frac{ds}{(2a)^{\frac{1}{2}}};$$

it is evident that the centre of gravity will be in the axis of the cycloid; therefore $\bar{x} = \pi a$; and as k and ρ are constant, (2) becomes

$$\bar{y} = \frac{\int_0^{2a} \frac{y\,dy}{(2a - y)^{\frac{1}{2}}}}{\int_0^{2a} \frac{dy}{(2a - y)^{\frac{1}{2}}}} = \tfrac{4}{3}a.$$

Cor.—For the arc of a semi-cycloid, we get

$$\bar{x} = \tfrac{4}{3}a, \quad \bar{y} = \tfrac{4}{3}a.$$

4. Find the centre of gravity of a circular arc of uniform section, the density varying as the length of the arc from one extremity.

Let AB (Fig. 39), be the arc; let μ be the density at the units distance from A, then μs will be the density at the distance s from A: let OA be the axis of x, and α the $\angle$AOB. Then, putting μs for ρ, and $a \cos\theta$, $a \sin\theta$, $a\,d\theta$, and $a\theta$, for x, y, ds, and s, in (1) and (2),

$$\bar{x} = \frac{\int k \cdot \mu a\theta \cdot a \cos\theta \cdot a\, d\theta}{\int k \cdot \mu a\theta \cdot a\, d\theta} = a\frac{\int_0^\alpha \theta \cos\theta\, d\theta}{\int_0^\alpha \theta\, d\theta}$$

$$= 2a\frac{\alpha \sin\alpha + \cos\alpha - 1}{\alpha^2}.$$

$$\bar{y} = \frac{\int k \cdot \mu a\theta \cdot a \sin\theta \cdot a\, d\theta}{\int k \cdot \mu a\theta \cdot a\, d\theta} = a\frac{\int_0^\alpha \theta \sin\theta\, d\theta}{\int_0^\alpha \theta\, d\theta}$$

$$= 2a\frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2}.$$

Cor.—For a quadrant we get

$$\bar{x} = \frac{4a}{\pi^2}(\pi - 2), \quad \bar{y} = \frac{8a}{\pi^2}.$$

5. Find the centre of gravity of one-half of a loop of a lemniscate whose equation is $r^2 = a^2 \cos 2\theta$, l being the length of the half-loop.

Here $$\frac{dr}{-a^2 \sin 2\theta} = \frac{r\, d\theta}{a^2 \cos 2\theta} = \frac{ds}{a^2}; \therefore \text{ etc.}$$

Ans. $\bar{x} = \dfrac{a^2}{2^{\frac{1}{2}}l}$; $\bar{y} = a^2\dfrac{2^{\frac{1}{2}} - 1}{2^{\frac{1}{2}}l}$.

6. Find the centre of gravity of a straight rod, the density of which varies as the nth power of the distance of each point from one end.

Take the origin at this end, suppose the axis of x to coincide with the axis of the rod, and let l = the length of the rod.

Ans. $\bar{x} = \dfrac{n + 1}{n + 2}l$.

7. Find the centre of gravity of the arc of a semi-cardioid, its equation being

$$r = a(1 + \cos\theta).$$

Ans. The co-ordinates of the centre of gravity referred to the axis of the curve and a perpendicular through the cusp, as axes of x and y, are

$$\bar{x} = y = \tfrac{4}{5}a.$$

79. Centre of Gravity of a Plane Area.—Let ABCD be an area bounded by the ordinates, AC and BD, the curve AB whose equation is given, and the axis of x; it is required to find the centre of gravity of this area, the lamina (Art. 67) being supposed of uniform thickness and density. We divide the area into an infinite number of infinitesimal elements (Art. 77). Suppose this to be done by drawing ordinates to the curve. Let PM and QN be two consecutive ordinates, let (x, y) be the point, P, and let g be the centre of gravity of the trapezoid, MPQN, whose breadth is dx and whose parallel sides are y and $y + dy$. The area of this trapezoid is $y\,dx$, (Cal., Art. 184).

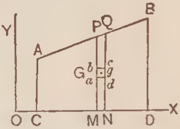

Fig. 40

Let ρ be the density and k the thickness of the lamina. Then (Art. 11) we have $dm = k\rho y\,dx$, which is the mass of the element MPQN; multiplying this mass by its co-ordinate, x, for example, we have the moment of the element $(k\rho\,xy\,dx)$, with respect to OY, and multiplying by the other co-ordinate, $\frac{1}{2}y$, we have the moment with respect to OX. Hence, substituting for dm in (1) and (2) of Art. 77, the surface element, $k\rho y\,dx$, and remembering that k and ρ are constants, we obtain, for the position of the centre of gravity of a body in the form of a plane area, the equations,

$$\bar{x} = \frac{\int xy\,dx}{\int y\,dx}; \quad \bar{y} = \tfrac{1}{2}\frac{\int y^2\,dx}{\int y\,dx}; \qquad (1)$$

the integrations extending over the whole area CABD.

EXAMPLES.

1. Find the centre of gravity of the area of a semi-parabola whose equation is $y^2 = 2px$.

Let $a =$ the axis, and b the extreme ordinate, then we have from (1)

$$x = \frac{\int_0^a \sqrt{2p}\,x^{\frac{3}{2}}\,dx}{\int_0^a \sqrt{2p}\,x^{\frac{1}{2}}\,dx} = \frac{\int_0^a x^{\frac{3}{2}}\,dx}{\int_0^a x^{\frac{1}{2}}\,dx} = \tfrac{3}{5}a;$$

$$y = \tfrac{1}{2}\frac{\int_0^a 2p\,x\,dx}{\int_0^a \sqrt{2p}\,x^{\frac{1}{2}}\,dx} = \sqrt{\frac{p}{2}}\frac{\int_0^a x\,dx}{\int_0^a x^{\frac{1}{2}}\,dx} = \tfrac{3}{8}b.$$

2. Find the centre of gravity of the area of an elliptic quadrant whose equation is

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Here $$\bar{x} = \frac{\int_0^a xy\,dx}{\int_0^a y\,dx} = \frac{\int_0^a \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}\,x\,dx}{\int_0^a \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}\,dx};$$

$$\therefore \quad \bar{x} = \frac{4a}{3\pi};$$

$$\bar{y} = \tfrac{1}{2}\frac{\int_0^a y^2\,dx}{\int_0^a y\,dx} = \tfrac{1}{2}\frac{\int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx}{\int_0^a \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}\,dx};$$

$$\therefore \quad \bar{y} = \frac{4b}{3\pi}.$$

Hence for the centre of gravity of the area of a circular quadrant we have

$$x = \bar{y} = \frac{4a}{3\pi}.$$

3. Find the centre of gravity of the area of a semicycloid.

Take the axis of the curve as axis of x, and a tangent at the highest point as axis of y; then the equation is (Anal. Geom., Art. 157),

$$y = a \operatorname{vers}^{-1}\frac{x}{a} + \sqrt{2ax - x^2};$$

where a is the radius of the generating circle. From (1) we have

$$\bar{x} = \frac{\int_0^{2a} xy\,dx}{\int_0^{2a} y\,dx} = \frac{\left[\frac{yx^2}{2} - \int \frac{x^2}{2}\,dy\right]_0^{2a}}{\left[yx - \int x\,dy\right]_0^{2a}}$$

$$= \frac{\frac{1}{2}\left[yx^2 - \int x\,(2ax - x^2)^{\frac{1}{2}}\,dx\right]_0^{2a}}{\left[yx - \int (2ax - x^2)^{\frac{1}{2}}\,dx\right]_0^{2a}} = \frac{1}{2}\,\frac{\pi a\,(2a)^2 - \frac{1}{2}\pi a^3}{\pi a \cdot 2a - \frac{1}{2}\pi a^2}$$

since when $x = 0$ and $2a$, $y = 0$ and πa.

$$\therefore \quad \bar{x} = \tfrac{7}{6}a.$$

Also,

$$\bar{y} = \frac{1}{2}\,\frac{\int_0^{2a} y^2\,dx}{\int_0^{2a} y\,dx} = \frac{\left[y^2x - 2\int yx\,dy\right]_0^{2a}}{3\pi a^2}$$

$$= \frac{\left[y^2x - 2\int y\,(2ax - x^2)^{\frac{1}{2}}\,dx\right]_0^{2a}}{3\pi a^2}$$

$$= \frac{\left[y^2x - 2a\int (2ax - x^2)^{\frac{1}{2}} \text{ vers}^{-1}\frac{x}{a}\,dx - 2\int (2ax - x^2)\,dx\right]_0^{2a}}{3\pi a^2}$$

$$= \frac{\left[y^2x - 2ax^2 + \tfrac{2}{3}x^3 - 2a\int (2ax - x^2)^{\frac{1}{2}} \text{ vers}^{-1}\frac{x}{a}\,dx\right]_0^{2a}}{3\pi a^2}$$

$$= \frac{2\pi^2a^3 - \dfrac{8a^3}{3} - \dfrac{\pi^2a^3}{2}}{3\pi a^2} = \frac{\tfrac{3}{2}\pi^2a^3 - \dfrac{8a^3}{3}}{3\pi a^2};$$

$$\therefore \quad y = \frac{a}{3\pi}\left(\tfrac{3}{2}\pi^2 - \tfrac{8}{3}\right),$$

which the student can verify by assuming

$$\text{vers}^{-1}\frac{x}{a} = \theta.$$

(See Todhunter's Statics, p. 118.)

80. Polar Elements of a Plane Area.—Let AB be the arc of a curve, and let it be required to find the centre of gravity of the area bounded by the arc AB and the extreme radii-vectors, OA and OB, drawn from the pole, O, to the extremities of the arc.

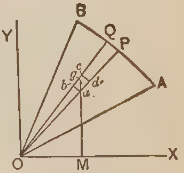

Fig. 41

Divide the area into infinitesimal triangles, such as POQ, included between two consecutive radii-vectors, OP and OQ. Let (r, θ) be the point, P, then the area of the element, POQ $= \frac{1}{2}r^2\,d\theta$ (Cal., Art. 191); and if the thickness and density of the lamina are uniform, the centre of gravity of this elementary triangle will be on a straight line drawn from O to the middle of PQ, and at a distance of two-thirds of this straight line from O (Art. 73). Hence the co-ordinates of the centre of gravity, g, of POQ, are OM and Mg, or,

$$\tfrac{2}{3}r\cos\theta, \quad \text{and} \quad \tfrac{2}{3}r\sin\theta.$$

Hence, (Art. 77),

$$\bar{x} = \frac{\int \frac{2}{3}r\cos\theta \cdot \frac{1}{2}r^2\,d\theta}{\int \frac{1}{2}r^2\,d\theta} = \frac{2}{3}\,\frac{\int r^3\cos\theta\,d\theta}{\int r^2\,d\theta}; \tag{1}$$

$$\bar{y} = \frac{\int \frac{2}{3}r\sin\theta \cdot \frac{1}{2}r^2\,d\theta}{\int \frac{1}{2}r^2\,d\theta} = \frac{2}{3}\,\frac{\int r^3\sin\theta\,d\theta}{\int r^2\,d\theta}; \tag{2}$$

the integrations extending over the whole area, AOB.

EXAMPLE.

Find the centre of gravity of the area of a loop of Bernouilli's Lemniscate whose equation is $r^2 = a^2\cos 2\theta$.

As the axis of the loop is symmetrical with respect to the axis of x, $\bar{y} = 0$, and the abscissa of the centre of gravity of the whole loop is evidently the same as that of the half-loop above the axis. Substituting in (1) for r its value $a\cos^{\frac{1}{2}} 2\theta$, we have

$$\bar{x} = \tfrac{2}{3}a\,\frac{\int_0^{\frac{\pi}{4}}\cos^{\frac{3}{2}} 2\theta\cos\theta\,d\theta}{\int_0^{\frac{\pi}{4}}\cos 2\theta\,d\theta}$$

$$= \tfrac{4}{3}a\int_0^{\frac{\pi}{4}}(1 - 2\sin^2\theta)^{\frac{3}{2}}\,d\sin\theta.$$

Put $\sin\theta = \dfrac{\sin\phi}{\sqrt{2}}$, then

$$\bar{x} = \frac{4a}{3\sqrt{2}} \int_0^{\frac{\pi}{2}} \cos^4 \phi \, d\phi = \frac{4a}{3\sqrt{2}} \cdot \tfrac{3}{8} \frac{\pi}{2} \text{ (Cal., Art. 157).}$$

$$\therefore \quad \bar{x} = \frac{\pi a}{4\sqrt{2}}.$$

81. Double Integration.—Polar Formulæ.—When the density of the lamina varies from point to point, it may be necessary to divide it into elements of the second order instead of rectangular or triangular elements of the first order (Arts. 79 and 80).

Suppose that the density of the lamina AOB (Fig. 41), is not uniform. If we divide it into triangular elements, POQ, the element of mass will be no longer proportional to the element of area, POQ $= \frac{1}{2}r^2 \, d\theta$; nor will the centre of gravity of the triangle, POQ, be $\frac{2}{3}r$ distant from O.

Let a series of circles be described with O as a centre, the distance between any two successive circles being dr. These circles will divide the triangle, POQ, into an infinite number of rectangular elements, $abcd = r d\theta \, dr$. If k is the thickness and ρ is the density of the lamina at this element, the element of mass will be $dm = k\rho r \, d\theta \, dr$; and the co-ordinates of its centre of gravity will be $r \cos \theta$ and $r \sin \theta$. Hence, from (1) and (2) of Art. 77, we have

$$\bar{x} = \frac{\int\int k\rho r \cos\theta \, r d\theta \, dr}{\int\int k\rho r \, d\theta \, dr} = \frac{\int\int k\rho r^2 \cos\theta \, d\theta \, dr}{\int\int k\rho r \, d\theta \, dr}; \qquad (1)$$

and

$$y = \frac{\int\int k\rho r^2 \sin\theta \, d\theta \, dr}{\int\int k\rho r \, d\theta \, dr}. \qquad (2)$$

In each of these integrals the values of k and ρ are to be substituted in terms of r and θ, and the integrations taken between proper limits.

EXAMPLE.

Find the centre of gravity of the area of a cardioid in which the density at a point increases directly as its distance from the cusp.

Let $\mu =$ the density at the unit's distance from the cusp, then $\rho = \mu r$, is the density at the distance r from the cusp.

As the axis of the curve is an axis of symmetry (Art. 67), $\bar{y} = 0$, and the abscissa of the whole curve is the same as for the half above the axis; then (1) becomes

$$x = \frac{\int_0^{\pi}\int_0^{r} r^3 \cos\theta \, d\theta \, dr}{\int_0^{\pi}\int_0^{r} r^2 \, d\theta \, dr}$$

$$= \frac{3}{4}\frac{\int_0^{\pi} r^4 \cos\theta \, d\theta}{\int_0^{\pi} r^3 \, d\theta},$$

by performing the r-integration.

The equation of the curve is

$$r = a\,(1 + \cos\theta) = 2a\cos^2\frac{\theta}{2}.$$

Substituting this value for r, and putting $\frac{\theta}{2} = \phi$, we have

$$\bar{x} = \frac{3}{2}a\frac{\int_0^{\frac{\pi}{2}} \cos^8\phi\,(2\cos^2\phi - 1)\,d\phi}{\int_0^{\frac{\pi}{2}} \cos^6\phi \, d\phi}$$

$$= \frac{21}{20}a.$$

82. Double Integration.—Rectangular Formulæ.—Let a series of consecutive straight lines be drawn parallel to the axes of x and y respectively, dividing the area, ABCD, (Fig. 40), into an infinite number of rectangular elements of the second order. Then the area of each element, as $abcd, = dx\,dy$; and if k and ρ are the thickness and density of the lamina at this element, the element of mass will be $dm = k\rho\,dx\,dy$; and the co-ordinates of its centre of gravity will be x and y. Hence from (1) and (2) of Art. 77, we have

$$\bar{x} = \frac{\int\int k\,\rho x\,dx\,dy}{\int\int k\rho\,dx\,dy}; \tag{1}$$

$$\bar{y} = \frac{\int\int k\,\rho y\,dx\,dy}{\int\int k\rho\,dx\,dy}. \tag{2}$$

the integrations being taken between proper limits.

EXAMPLE

Find the centre of gravity of the area of a cycloid the density of which varies as the nth power of the distance from the base.

Take the base as the axis of x and the starting point as the origin. Then the equation of the curve is

$$x = a \operatorname{vers}^{-1}\frac{y}{a} - (2ay - y^2)^{\frac{1}{2}};$$

$$\therefore\quad dx = \frac{y\,dy}{\sqrt{2ay - y^2}}.$$

Let $\rho = \mu y^n$ = density at the distance y from the base. It is evident that the centre of gravity will be in the axis of the cycloid; therefore $\bar{x} = \pi a$; and as k is constant (2) becomes

$$\bar{y} = \frac{\int_0^{2\pi a}\int_0^y y^{n+1}\,dy\,dx}{\int_0^{2\pi a}\int_0^y y^n\,dy\,dx}$$

$$= \frac{n+1}{n+2}\frac{\int_0^{2\pi a} y^{n+2}\,dx}{\int_0^{2\pi a} y^{n+1}\,dx}$$

$$= \frac{n+1}{n+2}\frac{\int_0^{2a}\frac{y^{n+3}\,dy}{\sqrt{2ay-y^2}}}{\int_0^{2a}\frac{y^{n+2}\,dy}{\sqrt{2ay-y^2}}}$$

$$= \frac{n+1}{n+2}\cdot\frac{2n+5}{n+3}\,a\,\frac{\int_0^{2a}\frac{y^{n+2}\,dy}{\sqrt{2ay-y^2}}}{\int_0^{2a}\frac{y^{n+2}\,dy}{\sqrt{2ay-y^2}}};$$

$$\therefore\quad \bar{y} = \frac{n+1}{n+2}\cdot\frac{2n+5}{n+3}\,a.$$

83. Centre of Gravity of a Surface of Revolution.—Let a surface be generated by the revolution of the curve, AB (Fig. 40), round the axis of x. Then the elementary arc, PQ, $(= ds)$, generates an element of the surface whose area $= 2\pi y\,ds$ (Cal., Art. 193). If k is the thickness and ρ the density of the lamina or shell in this elementary zone, the element of mass will be $dm = 2\pi k\rho y\,ds$. Also the centre of gravity of this zone is in the axis of x at the point M whose abscissa is x and ordinate 0. Hence (1) of Art. 77 becomes, after cancelling 2π,

$$\bar{x} = \frac{\int k\rho xy\,ds}{\int k\rho y\,ds} \tag{1}$$

the integrations being taken between proper limits.

EXAMPLES.

1. Find the centre of gravity of the surface formed by the revolution of a semi-cycloid round its base.

The equation of the generating curve is

$$x = a \operatorname{vers}^{-1}\frac{y}{a} - \sqrt{2ay - y^2};$$

$$\therefore \quad \frac{dx}{y} = \frac{dy}{\sqrt{2ay - y^2}} = \frac{ds}{\sqrt{2ay}};$$

or

$$ds = \frac{\sqrt{2a}\,dy}{\sqrt{2a - y}}.$$

which in (1) gives, after cancelling $\sqrt{2a}\,k\rho$,

$$\bar{x} = \frac{\int_0^{2a} \frac{xy\,dy}{\sqrt{2a - y}}}{\int_0^{2a} \frac{y\,dy}{\sqrt{2a - y}}} = \tfrac{26}{15}a.$$

2. Find the centre of gravity of the surface formed by the revolution of a semi-cycloid round its axis.

It is clear that the centre of gravity lies on the axis of the curve; hence $\bar{y} = 0$.

The equation of the generating curve is

$$y = a \operatorname{vers}^{-1}\frac{x}{a} + \sqrt{2ax - x^2}.$$

Here
$$dy = \sqrt{\frac{2a - x}{x}}\, dx,$$

$$ds = \sqrt{2a}\, x^{-\frac{1}{2}}\, dx,$$

which in (1) gives

$$\bar{x} = \frac{\int_0^{2a} y x^{\frac{1}{2}}\, dx}{\int_0^{2a} y x^{-\frac{1}{2}}\, dx}$$

$$= \frac{\left[\frac{2}{3} y x^{\frac{3}{2}} - \frac{2}{3}\int x^{\frac{3}{2}}\, dy\right]_0^{2a}}{\left[2 y x^{\frac{1}{2}} - 2\int x^{\frac{1}{2}}\, dy\right]_0^{2a}}$$

$$= \frac{\left[\frac{2}{3} y x^{\frac{3}{2}} - \frac{2}{3}\int x\sqrt{2a - x}\, dx\right]_0^{2a}}{\left[2 y x^{\frac{1}{2}} - 2\int \sqrt{2a - x}\, dx\right]_0^{2a}}$$

$$= \frac{\frac{2}{3}\pi a\,(2a)^{\frac{3}{2}} - \frac{8}{45}\,(2a)^{\frac{5}{2}}}{2\pi a\,(2a)^{\frac{1}{2}} - \frac{4}{3}\,(2a)^{\frac{3}{2}}}$$

$$= \tfrac{2}{15}\, a\, \frac{15\pi - 8}{3\pi - 4}.$$

3. Find the centre of gravity of the surface formed by the revolution of the semi-cycloid round the axis of y in the last example, *i. e.*, round the tangent to the curve at the highest point.

Ans. $\bar{y} = \frac{a}{15}(15\pi - 8).$

84. Centre of Gravity of Any Curved Surface.— Let there be a shell having any given curved surface for one of its boundaries; and let $k =$ the thickness, $\rho =$ the density, and $ds =$ the area of an element of the surface at the point (x, y, z); then (1) of Art. 83 becomes

$$\bar{x} = \frac{\int k\rho x\, ds}{\int k\rho\, ds} \tag{1}$$

and similar expressions for $\bar{y}$ and $\bar{z}$.

Substituting the value of ds (Cal., Art. 201) and cancelling k and ρ, we have

$$\bar{x} = \frac{\int\int x\left(1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}\right)^{\frac{1}{2}} dx\, dy}{\int\int \left(1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}\right)^{\frac{1}{2}} dx\, dy}.$$

EXAMPLES.

1. Find the centre of gravity of one-eighth of the surface of a sphere.

Here $$x^2 + y^2 + z^2 = a^2,$$

$$\left(1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}\right)^{\frac{1}{2}} = \frac{a}{(a^2 - x^2 - y^2)^{\frac{1}{2}}}.$$

$$\therefore\ \bar{x} = \frac{\int\int \frac{x\, dx\, dy}{(a^2 - x^2 - y^2)^{\frac{1}{2}}}}{\int\int \frac{dx\, dy}{(a^2 - x^2 - y^2)^{\frac{1}{2}}}}.$$

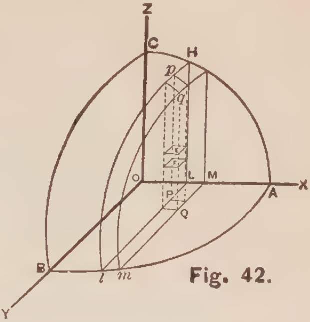

Fig. 42.

First perform the y-integration, x being constant, from $y = 0$ to $y = Ll = y_1 = \sqrt{a^2 - x^2}$; the effect will be to sum up all the elements similar to pq from H to l. The effect of a subsequent x-integration will be to sum all these elemental strips that are comprised in the surface of which OAB is the projection, and the limits of this integration are $x = 0$ and $x = OA = a$. Hence

$$\bar{x} = \frac{\displaystyle\int_0^a \int_0^{y_1} \frac{x\,dx\,dy}{(a^2 - x^2 - y^2)^{\frac{1}{2}}}}{\displaystyle\int_0^a \int_0^{y_1} \frac{dx\,dy}{(a^2 - x^2 - y^2)^{\frac{1}{2}}}}$$

$$= \frac{\displaystyle\int_0^a \tfrac{1}{2}\pi x\,dx}{\displaystyle\int_0^a \tfrac{1}{2}\pi\,dx} = \tfrac{1}{2}a.$$

Similarly $\qquad \bar{y} = \tfrac{1}{2}a, \quad \bar{z} = \tfrac{1}{2}a.$

2. Find the centre of gravity of one-eighth of the surface of the sphere if the density varies as the z-ordinate to any point of it. Here $\rho = \mu z$.

$$\textit{Ans.}\ \bar{x} = \frac{4a}{3\pi};\ \bar{y} = \frac{4a}{3\pi};\ \bar{z} = \frac{2a}{3}.$$

85. Centre of Gravity of a Solid of Revolution.—Let a solid be generated by the revolution of the curve, AB, (Fig. 40), round the axis of x. Then the elementary rectangle, $PQNM$, ($= y\,dx$), generates an element of the solid whose volume $= \pi y^2\, dx$ (Cal., Art. 203). Hence if the density of the solid is uniform, we have for the position of the centre of gravity (which evidently is in the axis of x),

$$\bar{x} = \frac{\int \pi y^2 x\, dx}{\int \pi y^2\, dx} = \frac{\int y^2 x\, dx}{\int y^2\, dx}. \qquad (1)$$

the integrations being extended over the whole area, CABD, of the bounding curve.

If the density varies, the element of mass may require to be taken differently. If the density varies with x alone, *i. e.*, if it is uniform all over the rectangular strip, $PQNM$, the volume may be divided up as already done, and the element of mass $= \pi \rho y^2\, dx$. Hence, we shall have in this case,

$$\bar{x} = \frac{\int \rho y^2 x\, dx}{\int \rho y^2\, dx}. \qquad (2)$$

If the density varies as y alone, we may take a rectangular element of area of the second order, $dx\, dy$, at the point (x, y); this area will generate an element of volume $= 2\pi y\, dx\, dy$; therefore the element of mass $= 2\pi \rho y\, dx\, dy$, and we have

$$\bar{x} = \frac{\int\int \rho x y\, dx\, dy}{\int\int \rho y\, dx\, dy}; \qquad (3)$$

the y-integrations being performed first, from 0 to y, the ordinate of a point P, on the bounding curve; and then the x-integrations from OC to OD.

EXAMPLES.

1. Find the centre of gravity of the hemisphere generated by the revolution of the quadrant, AD, (Fig. 39), round OA (taken as axis of x), (1) when the density is uniform; (2) when it is constant over a section perpendicular to OA and varies as the distance of this section from OD; (3) when it is constant at the same distance from OA and varies as this distance.

(1) From (1) we have

$$\bar{x} = \frac{\int y^2 x \, dx}{\int y^2 \, dx}.$$

Putting $x = r \cos \theta$, and $y = r \sin \theta$, where r is the radius of the circle and integrating between $\theta = 0$ and $\theta = \frac{\pi}{2}$, we have

$$\bar{x} = \tfrac{3}{8} r.$$

(2) Since $\rho = \mu x$, we have from (2)

$$x = \frac{\int x^2 y^2 \, dx}{\int x y^2 \, dx},$$

which gives $$x = \tfrac{8}{15} r.$$

(3) Since $\rho = \mu y$, we have from (3)

$$\bar{x} = \frac{\int \int x y^2 \, dx \, dy}{\int \int y^2 \, dx \, dy} = \frac{\int x y^3 \, dx}{\int y^3 \, dz}.$$

and the previous substitutions for x and y give

$$\bar{x} = \frac{16r}{15\pi}.$$

2. Find the centre of gravity of a paraboloid of revolution, the length of whose axis is h. *Ans.* $\bar{x} = \frac{2}{3}h$.

3. Find the centre of gravity (1) of a portion of a prolate spheroid, the length of whose axis measured from the vertex is c, and (2) of a hemi-spheroid.

Ans. (1) $\bar{x} = \frac{c}{4}\frac{8a - 3c}{3a - c}$; (2) $\bar{x} = \frac{5}{8}a$.

86. Polar Formulæ.—Let a solid be generated by the revolution of AB, (Fig. 41), round the axis of x. Then the elementary rectangle, *abcd*, whose mass $= \rho r\, d\theta\, dr$, (Art. 81), the thickness being omitted, generates a ring which is an element of the solid whose volume $= 2\pi r \sin\theta\, \rho r\, d\theta\, dr$; and the abscissa of the centre of gravity of the ring is $r\cos\theta$. Hence (1) of Art. 77 becomes

$$x = \frac{\int\int \rho r^3 \sin\theta \cos\theta\, d\theta\, dr}{\int\int \rho r^2 \sin\theta\, d\theta\, dr}. \qquad (1)$$

in which ρ must be a function of r and θ in order that the integrations may be effected.

If the density depends only on the distance from a fixed point in the axis of revolution, this point may be taken as origin, and ρ will be a function of r; if the density depends only on the distance from the axis of revolution, ρ will be a function of $r \sin\theta$.

EXAMPLE.

The vertex of a right circular cone is in the surface of a sphere, the axis of the cone coinciding with a diameter of the sphere, the base of the cone being a portion of the surface of the sphere. Find the distance of the centre of gravity of the cone from its vertex, 2α being its vertical angle, and a, the radius of the sphere.

Here the r-limits are 0 and $2a\cos\theta$; the θ-limits are 0 and α; ρ is constant; hence from (1) we have

$$\bar{x} = \frac{\int_0^\alpha \int_0^r r^3 \sin\theta \cos\theta \, d\theta \, dr}{\int_0^\alpha \int_0^r r^2 \sin\theta \, d\theta \, dr}$$

$$= \tfrac{3}{4} \frac{\int_0^\alpha (2a\cos\theta)^4 \sin\theta \cos\theta \, d\theta}{\int_0^\alpha (2a\cos\theta)^3 \sin\theta \, d\theta}$$

$$= \tfrac{3}{2} a \frac{\int_0^\alpha \cos^5\theta \sin\theta \, d\theta}{\int_0^\alpha \cos^3\theta \sin\theta \, d\theta}$$

$$= \frac{1 - \cos^6\alpha}{1 - \cos^4\alpha} a.$$

87. Centre of Gravity of any Solid.—Let (x, y, z) and $(x + dx, y + dy, z + dz)$ be two consecutive points E and F, (Fig. 42), within the solid whose centre of gravity is to be found. Through E, pass three planes parallel to the co-ordinate planes xy, yz, zx; also through F pass three planes parallel to the first. The solid included by these six planes is an infinitesimal parallelopiped, of which E and F are two opposite angles, and the volume $= dx\, dy\, dz$. If ρ is the density of the body at E, the element of mass at E $= \rho\, dx\, dy\, dz$. Hence the co-ordinates of the centre of gravity of the solid are given by the equations

$$\bar{x} = \frac{\int\int\int \rho x \, dx \, dy \, dz}{\int\int\int \rho \, dx \, dy \, dz}, \tag{1}$$

$$\bar{y} = \frac{\int\int\int \rho y \, dx \, dy \, dz}{\int\int\int \rho \, dx \, dy \, dz}, \tag{2}$$

$$\bar{z} = \frac{\int\int\int \rho z \, dx \, dy \, dz}{\int\int\int \rho \, dx \, dy \, dz}, \tag{3}$$

the integrations being extended over the whole solid.

EXAMPLES.

1. Find the centre of gravity of the eighth part of an ellipsoid included between its three principal planes.*

Let the equation of the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Here the limits of the z-integration are

$$c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} \text{ and } 0,$$

which call z_1 and 0; the limits of y are

$$Ll = b\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} \text{ and } 0,$$

which call y_1 and 0; the x-limits are a and 0.

---

* Planes of xy, yz, zx.

First integrate with respect to z, and we obtain the infinitesimal prismatic column whose base is PQ, (Fig. 42), and whose height is Pp. Then we integrate with respect to y, and obtain the sum of all the columns which form the elemental slice H$plmq$. Then integrating with respect to x, we obtain the sum of all the slices included in the solid, OABC. Hence (1) becomes, since the density is uniform,

$$\bar{x} = \frac{\int_0^a \int_0^{y_1} \int_0^{z_1} x\,dx\,dy\,dz}{\int_0^a \int_0^{y_1} \int_0^{z_1} dx\,dy\,dz}$$

$$= \frac{\int_0^a \int_0^{y_1} x\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} dx\,dy}{\int_0^a \int_0^{y_1} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} dx\,dy}$$

$$= \frac{\int_0^a \left(1 - \frac{x^2}{a^2}\right) x\,dx}{\int_0^a \left(1 - \frac{x^2}{a^2}\right) dx}$$

$$\therefore \quad \bar{x} = \tfrac{3}{8}a.$$

Similarly $\quad \bar{y} = \tfrac{3}{8}b, \quad z = \tfrac{3}{8}c.$

2. Find the centre of gravity of the solid bounded by the planes $z = \beta x$, $z = \gamma x$, and the cylinder $y^2 = 2ax - x^2$.

Ans. $\bar{x} = \tfrac{5}{4}a;\ \bar{y} = 0;\ \bar{z} = \frac{5a}{8}(\beta + \gamma).$

88. Polar Elements of Mass.—Let Fig. 43 represent the portion of the volume of a solid included between its bounding surface and three rectangular co-ordinate planes.

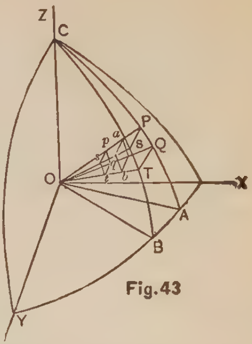

Fig. 43

(1) Through the axis of z draw a series of consecutive planes, dividing the solid into wedge-shaped slices such as COBA.

(2) Round the axis of z describe a series of right cones with their vertices at O, thus dividing each slice into elementary pyramids like O–PQST.

(3) With O as a centre describe a series of consecutive spheres; thus the solid is divided into elementary rectangular parallelopipeds similar to $abpt$, whose volume $= ap \cdot ps \cdot st$.

Let $\quad \text{XOA} = \phi, \quad \text{COP} = \theta, \quad \text{O}p = r,$

$$\text{AOB} = d\phi, \quad \text{POQ} = d\theta, \quad pa = dr.$$

Then pq is the arc of a circle whose radius is r, and the angle is $d\theta$; therefore

$$pq = r d\theta.$$

Also ps is the arc of a circle in which the angle is $d\phi$, and the radius is the perpendicular from p on OZ, or $r \sin \theta$; therefore

$$ps = r \sin \theta \, d\phi.$$

Therefore the volume of the elementary parallelopiped =

$$r^2 \sin \theta \, dr \, d\theta \, d\phi;$$

and if ρ is the density of the solid at p, the element of mass is

$$\rho r^2 \sin \theta \, dr \, d\theta \, d\phi.$$

Also the co-ordinates of the centre of gravity of this element are

$$r \sin \theta \cos \phi, \quad r \sin \theta \sin \phi, \quad \text{and} \quad r \cos \theta;$$

hence for the centre of gravity of the whole solid we have

$$\bar{x} = \frac{\int\int\int \rho r^3 \sin^2\theta \cos\phi \, dr \, d\theta \, d\phi}{\int\int\int \rho r^2 \sin\theta \, dr \, d\theta \, d\phi};$$

$$\bar{y} = \frac{\int\int\int \rho r^3 \sin^2\theta \sin\phi \, dr \, d\theta \, d\phi}{\int\int\int \rho r^2 \sin\theta \, dr \, d\theta \, d\phi};$$

$$\bar{z} = \frac{\int\int\int \rho r^3 \sin\theta \cos\theta \, dr \, d\theta \, d\phi}{\int\int\int \rho r^2 \sin\theta \, dr \, d\theta \, d\phi}.$$

the limits of integration being determined by the figure of the solid considered.

The angles, θ and ϕ, are sometimes called the *co-latitude,* and *longitude,* respectively.

EXAMPLES.

1. Find the centre of gravity of a hemisphere whose density varies as the nth power of the distance from the centre.

Take the axis of z perpendicular to the plane base of the hemisphere. Let a = the radius of the sphere, and $\rho = \mu r^n$, where μ is the density at the units distance from the centre. First integrate with respect to r from 0 to a, and we obtain the infinitesimal pyramid O–PQST. Then integrate with respect to θ from 0 to $\frac{1}{2}\pi$, and we obtain the sum of all the pyramids which form the elemental slice, COBA. Then integrating with respect to ϕ from 0 to 2π, we obtain the sum of all the slices included in the hemisphere, Hence,

$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^a r^{n+3}\sin\theta\cos\theta\,dr\,d\theta\,d\phi}{\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^a r^{n+2}\sin\theta\,dr\,d\theta\,d\phi}$$

$$= \frac{n+3}{n+4}a\,\frac{\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\sin\theta\cos\theta\,d\theta\,d\phi}{\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\sin\theta\,d\theta\,d\phi};$$

$$\therefore\quad \bar{z} = \frac{n+3}{n+4}\cdot\frac{a}{2};$$

and it is clear that $\bar{x} = \bar{y} = 0.$

2. Find the centre of gravity of a portion of a solid sphere contained in a right cone whose vertex is the centre of the sphere, the density of the solid varying as the nth power of the distance from the centre, the vertical angle of the cone being $= 2\alpha$, and the radius $= a$.

Take the axis of the cone as that of z, and any plane through it as that from which longitude is measured.

Ans. $\bar{z} = \frac{n+3}{n+4}\frac{a}{2}(1+\cos\alpha)$, and $\bar{x} = \bar{y} = 0.$

89. Special Methods.—In the preceding Articles we have given the usual formulæ for finding the centres of gravity of bodies, but particular cases may occur which may be most conveniently treated by special methods.

EXAMPLES.

1. A circle revolves round a tangent line through an angle of 180°; find the centre of gravity of the solid generated.

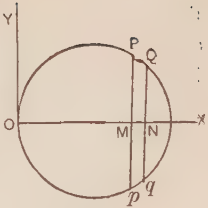

Fig. 44

Let OY be the tangent line about which the circle revolves, and let the plane of the paper bisect the solid; the centre of gravity will therefore lie in the axis of x. Let P and Q be two consecutive points; and let OM $= x$, and MP $= y = \sqrt{2ax - x^2}$. The elementary rectangle, PQqp, will generate a semi-cylindrical shell, whose volume $= 2y\,\pi x\,dx$, the centre of gravity of which will be in the axis of x at a distance $\frac{2x}{\pi}$ from O (Art 78, Ex. 1, Cor.). Hence,

$$\bar{x} = \frac{\int_0^{2a} \frac{2x}{\pi} 2y\,\pi x\,dx}{\int_0^{2a} 2y\,\pi x\,dx}$$

$$= \frac{2}{\pi}\,\frac{\int_0^{2a} x^2\sqrt{2ax - x^2}\,dx}{\int_0^{2a} x\sqrt{2ax - x^2}\,dx} = \frac{5a}{2\pi}.$$

2. Find the centre of gravity of a right pyramid of uniform density, whose base is any regular plane figure.

Let the vertex of the pyramid be the origin, and the axis of the pyramid the axis of x; divide the pyramid into slices of the thickness dx by planes perpendicular to the axis. Then as the areas of these sections are as the squares of their homologous sides, and as the sides are as their distances from the vertex, so will the areas of the sections be as the squares of their distances from the vertex, and therefore the masses of the slices are as the squares of their distances from the vertex. Now imagine each slice to be condensed into its centre of gravity, which point is on the axis of x. Then the problem is reduced to finding the centre of gravity of a material line in which the density varies as the square of the distance from one end, and which may be found as in Ex. 6, (Art. 78). Calling a the altitude of the pyramid, we have

$$\bar{x} = \frac{\int_0^a x^3\,dx}{\int_0^a x^2\,dx} = \tfrac{3}{4}a,$$

which is the same as in Art. 75.

90. Theorems of Pappus.*—(1) *If a plane curve revolve round any axis in its plane, the area of the surface generated is equal to the length of the revolving curve multiplied by the length of the path described by its centre of gravity.*

Let s denote the length of the curve, x, y, the co-ordinates of one of its points, $\bar{x}$, $\bar{y}$, the co-ordinates of the centre of gravity of the curve; then, if the curve is of constant thickness and density, we have from (2) of Art. 78,

$$y = \frac{\int y\,ds}{\int ds};$$

$$\therefore\ \ 2\pi\bar{y}s = 2\pi\int y\,ds; \tag{1}$$

the second member of which is the area of the surface generated by the revolution of the curve whose length is s about the axis of x, (Cal., Art. 193); and the first member is the length of the revolving curve, s, multiplied by the length of the path described by its centre of gravity, $2\pi\bar{y}$.

---

* Usually called Guldin's Theorems, but originally enunciated by Pappus. (See Walton's Mechanical Problems, p. 42, 3d Ed.)

(2) *If a plane area revolve round any axis in its plane, the volume generated is equal to the area of the revolving figure multiplied by the length of the path described by its centre of gravity.*

Let A denote the plane area, and let it be of constant thickness and density, then (2) of Art. 82 becomes

$$\bar{y} = \frac{\int\int y\,dx\,dy}{\int\int dx\,dy};$$

or $$2\pi\bar{y}\int\int d\text{A} = 2\pi\int\int y\,dx\,dy,$$

(substituting dA for $dx\,dy$),

$$\therefore\quad 2\pi\,\bar{y}\text{A} = \pi\int y^2\,dx, \tag{2}$$

the integral being taken for every point in the perimeter of the area; but the second member is the volume of the solid generated by the revolution of the area (Cal., Art. 203); and the first member is the area of the revolving figure, A, multiplied by the length of the path described by its centre of gravity, $2\pi\bar{y}$.

Cor.—If the curve or area revolve through any angle, θ, instead of 2π, (1) and (2) become

$$\theta\,\bar{y}s = \theta\int y\,ds, \tag{3}$$

and $$\theta\,\bar{y}\text{A} = \tfrac{1}{2}\theta\int y^2\,dx, \tag{4}$$

and the theorems are still true.

Sch.—If the axis cuts the revolving curve or area, the theorems still apply with the convention that the surface or volume generated by the portions of the curve or area on opposite sides of the axis are affected with opposite signs.

EXAMPLES.

1. A circle of radius, a, revolves round an axis in its own plane at a distance, c, from its centre; find the surface of the ring generated by it.

The length (circumference) of the revolving curve $= 2\pi a$; the length of the path described by its centre of gravity $= 2\pi c$;

$$\therefore \quad \text{the area of the surface of the ring} = 4\pi^2 ac.$$

2. An ellipse revolves round an axis in its own plane, the perpendicular distance of which from the centre is c; find the volume of the ring generated during a complete revolution.

Let a and b be the semi-axes of the ellipse; then the revolving area $= \pi ab$; the length of the path described by its centre of gravity $= 2\pi c$;

$$\therefore \quad \text{the volume of the ring} = 2\pi^2 abc.$$

Observe that the volume is the same for any position of the axes of the ellipse with respect to the axis of revolution, provided the perpendicular distance from that axis to the centre of the ellipse is the same.

3. The surface of a sphere, of radius a, $= 4\pi a^2$; the length of a semi-circumference $= \pi a$; find the length of the ordinate to the centre of gravity of the arc of a semicircle.

Ans. $\bar{y} = \dfrac{2a}{\pi}$.

4. The volume of a sphere, of radius a, $= \frac{4}{3}\pi a^3$; the area of a semicircle $= \frac{1}{2}\pi a^2$; find the distance of the centre of gravity of the semicircle from the diameter.

Ans. $\bar{y} = \dfrac{4a}{3\pi}$.

5. A circular tower, the diameter of which is 20 ft., is being built, and for every foot it rises it inclines 1 in. from

the vertical; find the greatest height it can reach without falling. *Ans.* 240 ft.

6. A circular table weighs 20 lbs. and rests on four legs in its circumference forming a square; find the least vertical pressure that must be applied at its edge to overturn it.

Ans. $20(\sqrt{2}+1) = 48.28$ lbs.

7. If the sides of a triangle be 3, 4, and 5 feet, find the distance of the centre of gravity from each side.

Ans. $\frac{4}{3}$, 1, $\frac{4}{5}$ ft.

8. An equilateral triangle stands vertically on a rough plane; find the ratio of the height to the base of the plane when the triangle is on the point of overturning.

Ans. $\sqrt{3} : 1$.

9. A heavy bar 14 feet long is bent into a right angle so that the lengths of the portions which meet at the angle are 8 feet and 6 feet respectively; show that the distance of the centre of gravity of the bar so bent from the point of the bar which was the centre of gravity when the bar was straight, is $\frac{9\sqrt{2}}{7}$ feet.

10. An equilateral triangle rests on a square, and the base of the triangle is equal to a side of the square; find the centre of gravity of the figure thus formed.

Ans. At a distance from the base of the triangle equal to $\frac{3}{8+2\sqrt{3}}$ of the base.

11. Find the inclination of a rough plane on which half a regular hexagon can just rest in a vertical position without overturning, with the shorter of its parallel sides in contact with the plane. *Ans.* $3\sqrt{3} : 5$.

12. A cylinder, the diameter of which is 10 ft., and height 60 ft., rests on another cylinder the diameter of which is

18 ft., and height 6 ft.; and their axes coincide; find their common centre of gravity. *Ans.* $27\frac{306}{331}$ ft. from the base.

13. Into a hollow cylindrical vessel 11 ins. high, and weighing 10 lbs., the centre of gravity of which is 5 ins. from the base, a uniform solid cylinder 6 ins. long and weighing 20 lbs., is just fitted; find their common centre of gravity. *Ans.* $3\frac{2}{3}$ ins. from base.

14. The middle points of two adjacent sides of a square are joined and the triangle formed by this straight line and the edges is cut off; find the centre of gravity of the remainder of the square.

Ans. $\frac{1}{21}$ of diagonal from centre.

15. A trapezoid, whose parallel sides are 4 and 12 ft. long, and the other sides each equal to 5 ft., is placed with its plane vertical, and with its shortest side on an inclined plane; find the relation between the height and base of the plane when the trapezoid is on the point of falling over.

Ans. 8 : 7.

16. A regular hexagonal prism is placed on an inclined plane with its end faces vertical; find the inclination of the plane so that the prism may just tumble down the plane.

Ans. 30°.

17. A regular polygon just tumbles down an inclined plane whose inclination is 10°; how many sides has the polygon? *Ans.* 18.

18. From a sphere of radius R is removed a sphere of radius r, the distance between their centres being c; find the centre of gravity of the remainder.

Ans. It is on the line joining their centres, and at a distance $\frac{cr^3}{R^3 - r^3}$ from the centre.

19. A rod of uniform thickness is made up of equal lengths of three substances, the densities of which taken in

order are in the proportion of 1, 2, and 3 ; find the position of the centre of gravity of the rod.

Ans. At $\frac{7}{18}$ of the whole length from the end of the densest part.

20. A heavy triangle is to be suspended by a string passing through a point on one side ; determine the position of the point so that the triangle may rest with one side vertical.

Ans. The distance of the point from one end of the side = twice its distance from the other end.

21. The sides of a heavy triangle are 3, 4, 5, respectively ; if it be suspended from the centre of the inscribed circle show that it will rest with the shortest side horizontal.

22. The altitude of a right cone is h, and a diameter of the base is b ; a string is fastened to the vertex and to a point on the circumference of the circular base, and is then put over a smooth peg ; show that if the cone rests with its axis horizontal the length of the string is $\sqrt{(h^2 + b^2)}$.

23. Find the centre of gravity of the helix whose equations are

$$x = a \cos \phi ; \quad y = a \sin \phi ; \quad z = ka\phi.$$

$$\textit{Ans.}\ \bar{x} = ka\frac{y}{z};\ \bar{y} = ka\frac{a - x}{z};\ \bar{z} = \frac{z}{2}.$$

24. Find the distance of the centre of gravity of the catenary (Cal., Art. 177), from the axis of x, the curve being divided into two equal portions by the axis of y.

Ans. If $2l$ is the length of the curve and (h, k) is the extremity, the centre of gravity is on the axis of y at a distance $\frac{kl + ha}{2l}$ from the axis of x.

25. Find the centre of gravity of the area included between the arc of the parabola, $y^2 = 4ax$, and the straight line $y = kx$.

Ans. $\bar{x} = \frac{8a}{5k^2}$; $\bar{y} = \frac{2a}{k}$.

26. Find the centre of gravity of the area bounded by the cissoid and its asymptote, the equation of the cissoid being $y^2 = \frac{x^3}{2a - x}$.

Ans. $\bar{x} = \frac{5}{3}a$.

27. Find the centre of gravity of the area of the witch of Agnesi.

Ans. At a distance from the asymptote equal to $\frac{1}{4}$ of the diameter of the base circle.

28. Find the centre of gravity of the area included between the arc of a semi-cycloid, the circumference of the generating circle, and the base of the cycloid, the common tangent to the circle and cycloid at the vertex of the latter being taken as axis of x, the vertex being origin, and a the radius of the generating circle.

Ans. $\bar{x} = \frac{3\pi^2 - 8}{4\pi} a$; $\bar{y} = \frac{5}{4}a$.

29. Find the centre of gravity of the area contained between the curves $y^2 = ax$ and $y^2 = 2ax - x^2$, which is above the axis of x.

Ans. $\bar{x} = a\frac{15\pi - 44}{15\pi - 40}$; $\bar{y} = \frac{a}{3\pi - 8}$.

30. Find the centre of gravity of the area included by the curves $y^2 = ax$ and $x^2 = by$.

Ans. $\bar{x} = \frac{9}{20}a^{\frac{1}{3}}b^{\frac{2}{3}}$; $\bar{y} = \frac{9}{20}a^{\frac{2}{3}}b^{\frac{1}{3}}$.

31. Find the distance of the centre of gravity of the area of the circular sector, BOCA, (Fig. 39), from the centre.

Let $2\theta =$ the angle included by the bounding radii.

Ans. $\bar{x} = \frac{2}{3}a\frac{\sin\theta}{\theta}$.

32. Find the distance of the centre of gravity of the circular segment, BCA, (Fig. 39), from the centre.

$$\textit{Ans.}\ \bar{x} = \tfrac{2}{3} \cdot \frac{a \sin^3 \theta}{\theta - \sin \theta \cos \theta} = \frac{\overline{\mathrm{BC}}^3}{12 \text{ area of ABC}}.$$

33. Find the centre of gravity of the area bounded by the cardioid $r = a(1 + \cos \theta)$. *Ans.* $\bar{x} = \frac{5}{6}a$.

34. Find the centre of gravity of the area included by a loop of the curve $r = a \cos 2\theta$.

$$\textit{Ans.}\ \bar{x} = \frac{128a\sqrt{2}}{105\pi}.$$

35. Find the centre of gravity of the area included by a loop of the curve $r = a \cos 3\theta$.

$$\textit{Ans.}\ \bar{x} = \frac{81a\sqrt{3}}{80\pi}.$$

36. Find the centre of gravity of the area of the sector in Ex. 31, if the density varies directly as the distance from the centre.

$$\textit{Ans.}\ \bar{x} = \frac{3a}{4} \cdot \frac{\sin \theta}{\theta}.$$

37. Find the centre of gravity of the area of a circular sector in which the density varies as the nth power of the distance from the centre.

Ans. $\frac{n+2}{n+3} \cdot \frac{ac}{l}$, where a is the radius of the circle, l the length of the arc, and c the length of the chord, of the sector.

38. Find the centre of gravity of the area of a circle in which the density at any point varies as the nth power of the distance from a given point on the circumference.

Ans. It is on the diameter passing through the given point at a distance from this point equal to $\frac{2(n+2)}{n+4}a$, a being the radius.

39. Find the centre of gravity of the area of a quadrant of an ellipse in which the density at any point varies as the distance of the point from the major axis.

Ans. $\bar{x} = \frac{3}{8}a$; $\bar{y} = \frac{3\pi}{16}b$.

40. Find the distance of the centre of gravity of the surface of a cone from the vertex.

Let a = the altitude. *Ans.* $\bar{x} = \frac{2}{3}a$.

41. Find the centre of gravity of the surface formed by revolving the curve

$$r = a(1 + \cos\theta),$$

round the initial line. *Ans.* $\bar{x} = \frac{50a}{63}$.

42. A parabola revolves round its axis; find the centre of gravity of a portion of the surface between the vertex and a plane perpendicular to the axis at a distance from the vertex equal to $\frac{3}{4}$ of the latus rectum.

Ans. Its distance from the vertex $= \frac{29}{70}$ (latus rectum).

43. Find the centre of gravity of a cone, the density of each circular slice of which varies as the nth power of its distance from a parallel plane through the vertex.

Let the vertex be the origin and a the altitude.

Ans. $\bar{x} = \frac{n+3}{n+4}a$.

44. Find the centre of gravity of a cone, the density of every particle of which increases as its distance from the axis.

Ans. $\bar{x} = \frac{4}{5}a$, where the vertex is the origin and a the altitude.

45. Find the centre of gravity of the volume of uniform density contained between a hemisphere and a cone whose vertex is the vertex of the hemisphere and base is the base of the hemisphere.

Ans. $\bar{x} = \frac{a}{2}$, where the vertex is the origin and a the altitude.

46. Find the distance of the centre of gravity of a hemisphere from the centre, the radius being a.

Ans. $\bar{x} = \frac{3}{8}a$.

47. Find the centre of gravity of the solid generated by the revolution of the semicycloid,

$$y = \sqrt{2ax - x^2} + a \operatorname{vers}^{-1}\frac{x}{a},$$

(1) round the axis of x, and (2) round the axis of y.

Ans. (1) $\bar{x} = \frac{(63\pi^2 - 64)\,a}{6\,(9\pi^2 - 16)}$; (2) $\bar{y} = \left(\frac{16}{9} + \frac{\pi^2}{4}\right)\frac{2a}{\pi}$.

48. Find the centre of gravity of the volume formed by the revolution round the axis of x of the area of the curve

$$y^4 - axy^2 + x^4 = 0.$$

Ans. $\bar{x} = \frac{3a\pi}{32}$.

49. Find the centre of gravity of the volume generated by the revolution of the area in Ex. 29 round the axis of y.

Ans. $\bar{y} = \frac{5a}{2\,(15\pi - 44)}$.

50. Find the centre of gravity of a hemisphere when the density varies as the square of the distance from the centre.

Ans. $\bar{x} = \frac{5a}{12}$.

51. Find the centre of gravity of the solid generated by a semi-parabola bounded by the latus rectum, revolving round the latus rectum.

Ans. Distance from focus $= \frac{5}{32}$ of latus rectum.

52. The vertex of a right circular cone is at the centre of a sphere; find the centre of gravity of a body of uniform density contained within the cone and the sphere.

Ans. The distance of the centre of gravity from the vertex of the cone $= \frac{3a}{8}(1 + \cos \alpha)$, where $\alpha =$ the semi-vertical angle of the cone and $a =$ the radius of the sphere.

53. Find the distance from the origin to the centre of gravity of the solid generated by the revolution of the cardioid round its prime radius, its equation being

$$r = a(1 + \cos \theta).$$

Ans. $\bar{x} = \frac{4}{5}a$.

54. Find by Art. 90 (1) the surface and (2) the volume of the solid formed by the revolution of a cycloid round the tangent at its vertex.

Ans. Surface $= \frac{32}{3}\pi a^2$; Volume $= \pi^2 a^3$.

55. Find (1) the surface and (2) the volume of the solid formed by the revolution of a cycloid round its base.

Ans. (1) $\frac{64}{3}\pi a^2$; (2) $5\pi^2 a^3$.

56. An equilateral triangle revolves round its base, whose length is a; find (1) the area of the surface, and (2) the volume of the figure described.

Ans. (1) $\pi a^2 \sqrt{3}$; (2) $\frac{\pi a^3}{4}$.

57. Find (1) the surface and (2) the volume of a ring with a circular section whose internal diameter is 12 ins., and thickness 3 ins.

Ans. (1) 444.1 sq. in.; (2) 333.1 cub. in.

CHAPTER V.

FRICTION.

91. Friction.—*Friction is that force* which acts between two bodies at their surface of contact, and in the direction of a tangent to that surface, so as to resist their sliding on each other. It depends on the force with which the bodies are pressed together. All the curves and surfaces which we have hitherto considered were supposed to be smooth, and, as such, to offer no resistance to the motion of a body in contact with them in any other than a normal direction. Such curves and surfaces, however, are not to be found in nature. Every surface is capable of destroying a certain amount of force in its tangent plane, *i.e.*, it possesses a certain degree of *roughness*, in virtue of which it resists the sliding of other surfaces upon it. This resistance is called *friction*, and is of two kinds, viz., *sliding* and *rolling* friction. The first is that of a heavy body dragged on a plane or other surface, an axle turning in a fixed box, or a vertical shaft turning on a horizontal plate. Friction of the second kind is that of a wheel rolling along a plane. Both kinds of friction are governed by the same laws; the former is much greater than the latter under the same circumstances, and is the only one that we shall consider.

A *smooth* surface is one which opposes no resistance to the motion of a body upon it. A *rough* surface is one which does oppose a resistance to the motion of a body upon it.

The surfaces of all bodies consist of very small elevations and depressions, so that if they are pressed against each other, the elevations of one fit, more or less, into the depressions of the other, and the surfaces interpenetrate each other; and the mutual penetration is of course greater, if the pressing force is greater. Hence, when a force is applied so as to cause one body to move on another with which it is in contact, it is necessary, before motion can take place, either to break off the elevations or compress them, or force the bodies to separate far enough to allow them to pass each other. Much of this *roughness* may be removed by polishing; and the effect of much of it may be destroyed by lubrication.

Friction always acts along a tangent to the surface at the point of contact; and its direction is opposite to that of the line of motion; it presents itself in the motion of a body as a passive force or resistance,* since it can only *hinder* motion, but can never *produce* or *aid* it. In investigations in mechanics it can be considered as a force acting in opposition to every motion whose direction lies in the plane of contact of the two bodies. Whatever may be the direction in which we move a body resting upon a horizontal or inclined plane, the friction will always act in the opposite direction to that of the motion, *i. e.*, when we slide a body down an inclined plane, it will appear as a force up the plane. A surface may also resist sliding motion by means of the *adhesion* between its substance and that of another body in contact with it.†

The friction of a body on a surface is measured by the least force which will put the body in motion along the surface.

92. Laws of Friction.—In our ignorance of the constitution of bodies, the laws of friction must be deduced from experiment. Experiments made by Coulomb and Morin have established the following laws of friction:

(1) *The friction varies as the normal pressure when the materials of the surfaces in contact remain the same.* Subsequent experiments have, however, considerably modified this law, and shown that it can be regarded only as an approximation to the truth. When the pressure is very great it is found that the friction is less than this law would give.

---

* Weisbach, p. 309.

† See Rankine's Applied Mechanics, p. 209.

(2) *The friction is independent of the extent of the surfaces in contact so long as the normal pressure remains the same.* When the surfaces in contact are very small, as for instance a cylinder resting on a surface, this law gives the friction much too great.

These two laws are true when the body is on the point of moving, and also when it is actually in motion; but in the case of motion the magnitude of the friction is not always the same as when the body is *beginning* to move; when there is a difference, the friction is greater in the state *bordering* on motion than in actual motion.

(3) *The friction is independent of the velocity when the body is in motion.*

It follows from these laws that, if R be the normal pressure between the bodies, F the force of friction, and μ the constant ratio of the latter to the former *when slipping is about to ensue*, we have

$$F = \mu R. \tag{1}$$

The fraction μ is called the *co-efficient of friction;* and if the first law were true, μ would be strictly constant for the same pair of bodies, whatever the magnitude of the normal pressure between them might be. This, however, is not the case. When the normal pressure is nearly equal to that which would crush either of the surfaces in contact, the force of friction increases more rapidly than the normal pressure. Equation (1) is nevertheless very nearly true when the differences of normal pressure are not very great; and in what follows we shall assume this to be the case.

REMARK.—The laws of friction were established by Coulomb, a distinguished French officer of Engineers, and were founded on experiments made by him at Rochefort. The results of these experiments were presented in 1781 to the French Academy of Sciences, and in 1785 his Memoir on Friction was published. A very full abstract of this paper is given in De Young's *Natural Philosophy*, Vol. II, p. 170 (1st Ed.). Further experiments were made at Metz by Morin, 1831–34, by direction of the French military authorities, the result of which has been to confirm, with slight exceptions, all the results of Coulomb, and to determine with considerable precision the numerical values of the coefficients of friction, for all the substances usually employed in the construction of machines. (See Galbraith's Mechanics, p. 68, Twisden's Practical Mechanics, p. 138, and Weisbach's Mechanics, Vol. I, p. 317.)

93. Magnitudes of Coefficients of Friction.—Practically there is no observed coefficient much greater than 1. Most of the ordinary coefficients are less than $\frac{1}{2}$. The following results, selected from a table of coefficients,* will afford an idea of the amount of friction as determined by experiment ; these results apply to the friction of motion.

For iron on stone	μ	varies	between	.3 and .7.
For timber on timber	"	"	"	.2 and .5.
For timber on metals	"	"	"	.2 and .6.
For metals on metals	"	"	"	.15 and .25.

For full particulars on this subject the student is referred to Rankine's Applied Mechanics, p. 209, and Moseley's Engineering, p. 124, also to the treatise of M. Morin, where he will find the subject investigated in all its completeness.

94. Angle of Friction.—*The angle at which a rough plane or surface may be inclined so that a body, when acted upon by the force of gravity only, may just rest upon it without sliding, is called the Angle of Friction.*†

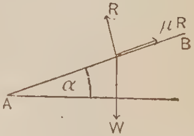

Fig. 45

Let α be the angle of inclination of the plane AB just as the weight is on the point of slipping down; W the weight of the body ; R the normal pressure on the plane; F the force of friction acting along the plane $= \mu R$ (Art. 92). Then, resolving the forces along and perpendicular to the plane we have for equilibrium

---

* Rankine's Applied Mechanics, p. 211.

† Sometimes called "the angle of repose;" also called "the limiting angle of resistance."

$$\mu R = W \sin \alpha ; \; R = W \cos \alpha ;$$

$$\therefore \quad \tan \alpha = \mu, \qquad (1)$$

which gives the limiting value of the inclination of the plane for which equilibrium is possible. The body will rest on the plane when the angle of inclination is less than the angle of friction, and will slide if the angle of inclination exceeds that angle; and this will be the case however great W may be; the reason being that in whatever manner we increase W, in the same proportion we increase the friction upon the plane, which serves to prevent W from sliding.

From (1) we see that the tangent of the angle of friction is equal to the coefficient of friction.

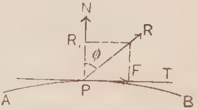

Fig. 46

95. Reaction of a Rough Curve or Surface.—Let AB be a rough curve or surface; P the position of a particle on it; and suppose the forces acting on P to be confined to the plane of the paper. Let R_1 = the normal resistance of the surface, acting in the normal, PN, and F = the force of friction, acting along the tangent, PT.

The resultant of R_1 and F, called the *Total Resistance** of the surface, is represented in magnitude and direction by the line $PR = R$, which is the diagonal of the parallelogram determined by R_1 and F. We have seen that the total resistance of a *smooth* surface is normal (Art. 11); but this limitation does not apply to a *rough* surface. Let ϕ denote the angle between R and the normal R_1; then ϕ is given by the equation

$$\tan \phi = \frac{F}{R_1}.$$

---

* Minchin's Statics, p. 54.

Hence, ϕ will be a maximum when the force of friction, F, bears the greatest ratio to the normal pressure R_1. But this greatest ratio is attained when the body is just on the point of slipping along the surface, and is what we called the coefficient of friction (Art. 92), that is

$$\frac{F}{R_1} = \mu;$$

$$\therefore \quad \tan\phi = \mu.$$

Therefore *the greatest angle by which the Total Resistance of a rough curve or surface can deviate from the normal is the angle whose tangent is the coefficient of friction for the bodies in contact; and this deviation is attained when slipping is about to commence.*

Cor.—By (1) of Art. 94, $\tan\alpha = \mu$;

$$\therefore \quad \phi = \alpha;$$

hence, the direction of the *total resistance, R,* is inclined at an angle α to the normal; *i. e., the greatest angle that the Total Resistance of a rough curve or surface can make with the normal is equal to the angle of friction, corresponding to the two bodies in contact.*

96. Friction on an Inclined Plane.—A body rests on a rough inclined plane, and is acted on by a given force, P, in a vertical plane which is perpendicular to the inclined plane; find the limits of the force, and the angle at which the least force capable of drawing the particle up the plane must act.

Let i = the inclination of the plane to the horizon; θ = the angle between the inclined plane and the line of action of P; μ = the coefficient of friction; and let us first suppose that the body is on the point of moving *down* the plane, so that friction is a force acting up the plane, then resolving along, and perpendicular to, the plane, we have

$$F + P\cos\theta = W\sin i,$$

$$R + P\sin\theta = W\cos i,$$

$$F = \mu R;$$

$$\therefore\quad P = W\frac{\sin i - \mu\cos i}{\cos\theta - \mu\sin\theta}. \qquad (1)$$

And if P is increased so that motion *up* the plane is just beginning, F acts in an opposite direction, and therefore the sign of μ must be changed and we have

$$P = W\frac{\sin i + \mu\cos i}{\cos\theta + \mu\sin\theta}. \qquad (2)$$

Hence, there will be equilibrium if the body be acted on by a force, the magnitude of which lies between the values of P in (1) and (2). Substituting $\tan\phi$ for μ (Art. 95); (2) becomes

$$P = W\frac{\sin(i + \phi)}{\cos(\phi - \theta)}. \qquad (3)$$

To determine θ in (2) so that P shall be a minimum we must put the first derivative of P with respect to $\theta = 0$, therefore

$$\frac{dP}{d\theta} = W(\sin i + \mu\cos i)\frac{\sin\theta - \mu\cos\theta}{(\cos\theta + \mu\sin\theta)^2} = 0;$$

$$\therefore\quad \tan\theta = \mu;$$

that is, the force P necessary to draw the body up the plane will be the least possible when $\theta =$ the angle of friction.

Hence we infer that a given force acts to the greatest advantage in dragging a weight up a hill, if the angle at which its line of action is inclined to the hill is equal to the angle of friction of the hill. Similarly, a force acts to the greatest advantage in dragging a weight along a horizontal plane if its line of action is inclined to the plane at the angle of friction of the plane. We may also determine from this the angle at which the traces of a drawing horse should be inclined to the plane of traction.

These results are those which are to be expected, because some part of the force ought to be expended in lifting the weight from the plane, so that friction may be diminished. (See Price's Anal. Mech's, Vol. I, p. 160.)

97. Friction on a Double-Inclined Plane.—Two bodies, whose weights are P and Q, rest on a rough double-inclined plane, and are connected by a string which passes over a smooth peg at a point, A, vertically over the intersection, B, of the two planes. Find the position of equilibrium.

Let α and β be the inclinations of the two planes; let $l =$ the length of the string, and $h =$ AB; and let θ and θ' be the angles the portions of the string make with the planes.

Suppose P is on the point of *ascending*, and Q of *descending*. Then, since the motion of each body is about to ensue, the total resistances, R and S, must each make the angle of friction with the corresponding normal (Art. 95, Cor.): and since the weight, P, is about to move upwards the friction must act downwards, and therefore R must lie below the normal, while, since Q is about to move downwards, the friction must act upwards, and therefore S must be above the normal.

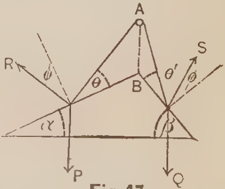

Fig. 47

If T is the tension of the string, we have for the equilibrium of P, (Art. 32),

$$T = P\frac{\sin(\alpha + \phi)}{\cos(\theta - \phi)}.$$

And for the equilibrium of Q,

$$T = Q\frac{\sin(\beta - \phi)}{\cos(\theta' + \phi)}.$$

Equating the values of T we get

$$P\frac{\sin(\alpha + \phi)}{\cos(\theta - \phi)} = Q\frac{\sin(\beta - \phi)}{\cos(\theta' + \phi)}, \tag{1}$$

and if P is about to move *down* the plane, the friction acts in an opposite direction, and therefore the sign of ϕ must be changed and we have

$$P\frac{\sin(\alpha - \phi)}{\cos(\theta + \phi)} = Q\frac{\sin(\beta + \phi)}{\cos(\theta' - \phi)}. \tag{2}$$

(1) or (2) is the only statical equation connecting the given quantities.

We obtain a geometric equation by expressing the length of the string in terms of h, α, β, θ, and θ', which is

$$l = h\left(\frac{\cos\alpha}{\sin\theta} + \frac{\cos\beta}{\sin\theta'}\right). \tag{3}$$

From (1) or (2) and (3) the values of θ and θ' can be found, and this determines the positions of P and Q.

Otherwise thus:

Instead of considering the total resistances, R and S, we may consider two *normal* resistances, R_1 and S_1, and two forces of friction, μR_1 and μS_1, acting respectively down the plane α and up the plane β. In this case, considering the equilibrium of P, and resolving forces along, and perpendicular to, the plane α, we have

$$\left.\begin{aligned} P \sin \alpha + \mu R_1 &= T \cos \theta, \\ P \cos \alpha &= R_1 + T \sin \theta, \end{aligned}\right\} \qquad (4)$$

and for the equilibrium of Q,

$$\left.\begin{aligned} Q \sin \beta &= \mu S_1 + T \cos \theta', \\ Q \cos \beta &= S_1 + T \sin \theta'. \end{aligned}\right\} \qquad (5)$$

Eliminating R_1, S_1, and T from (4) and (5) we get (1), the same statical equation as before.

The method of considering *total resistances* instead of their normal and tangential components is usually more simple than the separate consideration of the latter forces. (See Minchin's Statics, p. 60.)

Cor.—If Q is given and P be so small that it is about to ascend, its value, P_1, will be given by (1),

$$P_1 = Q \frac{\sin (\beta - \phi) \cos (\theta - \phi)}{\sin (\alpha + \phi) \cos (\theta' + \phi)}. \qquad (6)$$

and if P is so large that it is about to drag Q up, its value, P_2, will be given by (2)

$$P_2 = Q \frac{\sin (\beta + \phi) \cos (\theta + \phi)}{\sin (\alpha - \phi) \cos (\theta' - \phi)} \qquad (7)$$

the angles θ and θ' being connected by (3).

There will be equilibrium if Q be acted on by any force whose magnitude lies between P_1 and P_2.

98. Friction on Two Inclined Planes.—A beam rests on two rough inclined planes; find the position of equilibrium.

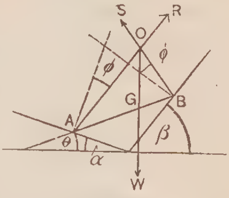

Fig. 48

Let a and b be the segments, AG and BG, of the beam; let θ be the inclination of the beam to the horizon, α and β the inclinations of the planes, and R and S the total resistances. Suppose that A is on the point of ascending; then the total resistances, R and S, must each make the angle of friction with the corresponding normal and act to the right of the normal.

The three forces, W, R, S, must meet in a point O (Art 62); and the angles GOA and GOB are equal to $\alpha + \phi$, and $\beta - \phi$, respectively.

Hence $\quad (a + b) \cot \text{BGO} = a \cot \text{GOA} - b \cot \text{GOB},$

or $\quad (a + b) \tan \theta = a \cot (\alpha + \phi) - b \cot (\beta - \phi). \quad (1)$

Cor.—If the planes are smooth, $\phi = 0$, and (1) becomes

$$(a + b) \tan \theta = a \cot \alpha - b \cot \beta.$$

(See Ex. 7, Art. 62.)

99. Friction of a Trunnion.*—*Trunnions are the cylindrical projections from the ends of a shaft, which rest on the concave surfaces of cylindrical boxes.* A shaft rests in a horizontal position, with its trunnions on rough cylindrical surfaces; find the resistance due to friction which is to be overcome when the shaft begins to turn about a horizontal axis.

---

* Sometimes called "Journal."

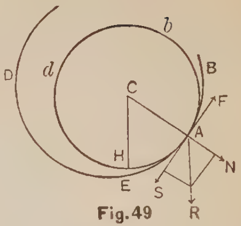

Fig. 49

Let bAd and BAED be two right sections of the trunnion and its box; the two circles are tangent to each other internally. If no rotation takes place the trunnion presses upon its lowest point, H, through which the direction of the resulting pressure, R, passes; if the shaft begins to rotate in the direction AH, the trunnion ascends along the inclined surface, EAB, in consequence of the friction on its bearing, until the force, S, tending to move it down just balances the friction, F. Resolving R into a normal force N and a tangential one, S, we have, since the tangential component of R in urging the trunnion down the surface = the friction which opposes it.

$$S = F = \mu N; \quad \text{but} \quad R^2 = S^2 + N^2;$$

or

$$R^2 = \mu^2 N^2 + N^2;$$

therefore

$$N = \frac{R}{\sqrt{1 + \mu^2}};$$

and the friction

$$F = \frac{\mu R}{\sqrt{1 + \mu^2}} = \frac{R \tan \phi}{\sqrt{1 + \tan^2 \phi}} \text{ (Art. 95)},$$

or

$$F = R \sin \phi.$$

Hence, *to find the friction upon a trunnion, multiply the resultant of the forces which act upon it by the sine of the angle of friction.*

100. Friction of a Pivot.—A heavy circular shaft rests in a vertical position, with its end, which is a circular section, on a horizontal plate; find the resistance due to friction which is to be overcome, when the shaft begins to revolve about a vertical axis.

Let a be the radius of the circular section of the shaft; let the plane of (r, θ) be the horizontal one of contact between the end of the shaft and the plate; and let the centre of the circular area of contact be the pole. Let $W =$ the weight of the shaft, then the vertical pressure on each unit of surface is $\frac{W}{\pi a^2}$; and therefore, if $r\,dr\,d\theta$ is the area-element, we have

$$\text{the pressure on the element} = \frac{W}{\pi a^2} r\,dr\,d\theta;$$

$$\therefore \quad \text{the friction of the element} = \mu \frac{W}{\pi a^2} r\,dr\,d\theta.$$

The friction is opposed to motion, and the direction of its action is tangent to the circle described by the element; the moment of the friction about the vertical axis through the centre

$$= \frac{\mu W r^2\,dr\,d\theta}{\pi a^2},$$

therefore the moment of friction of the whole circular end

$$= \int_0^{2\pi} \int_0^a \frac{\mu W r^2\,dr\,d\theta}{\pi a^2} = \frac{2\mu W a}{3}. \qquad (1)$$

and consequently varies as the radius. Hence arises the advantage of reducing to the smallest possible dimensions the area of the base of a vertical shaft revolving with its end resting on a horizontal bed.

From (1) we may regard the whole friction due to the pressure as acting at a single point, and at a distance from the centre of motion equal to two-thirds of the radius of the base of the shaft. This distance is called the mean lever of friction.

When the shaft is vertical, and rests upon its circular end in a cylindrical socket the cylindrical projection is called a Pivot.

EXAMPLES.

1. A mass whose weight is 750 lbs. rests on a horizontal plane, and is pulled by a force, P, whose direction makes an angle of 15° with the horizon; determine P and the total resistance, R, the coefficient of friction being .62.

Ans. $P = 413\cdot 3$ lbs.; $R = 756\cdot 9$ lbs.

2. Determine P in the last example if its direction is horizontal. *Ans.* $P = 465$ lbs.

3. Find the force along the plane required to draw a weight of 25 tons up a rough inclined plane, the coefficient of friction being $\frac{5}{12}$, and the inclination of the plane being such that 7 tons acting along the plane would support the weight if the plane were smooth.

Ans. Any force greater than 17 tons.

4. Find the force in the preceding example, supposing it to act at the most advantageous inclination to the plane.

Ans. $15\frac{9}{13}$ tons.

5. A ladder inclined at an angle of 60° to the horizon rests between a *rough* pavement and the *smooth* wall of a house. Show that if the ladder begin to slide when a man has ascended so that his centre of gravity is half way up, then the coefficient of friction between the foot of the ladder and the pavement is $\frac{1}{6}\sqrt{3}$.

6. A body whose weight is 20 lbs. is just sustained on a rough inclined plane by a horizontal force of 2 lbs., and a force of 10 lbs. along the plane; the coefficient of friction is $\frac{2}{5}$; find the inclination of the plane. *Ans.* $2\tan^{-1}(\frac{25}{48})$.

7. A heavy body is placed on a rough plane whose inclination to the horizon is $\sin^{-1}(\frac{3}{5})$, and is connected by a string passing over a smooth pulley with a body of equal weight, which hangs freely. Supposing that motion is on the point of ensuing up the plane, find the inclination of the string to the plane, the coefficient of friction being $\frac{1}{2}$.

Ans. $\theta = 2\tan^{-1}(\frac{1}{2})$.

8. A heavy body, acted upon by a force equal in magnitude to its weight, is just about to ascend a rough inclined plane under the influence of this force; find the inclination, θ, of the force to the inclined plane.

Ans. $\theta = \frac{\pi}{2} - i$, or $2\phi + i - \frac{\pi}{2}$, where $i =$ inclination of the plane, and $\phi =$ angle of friction. (θ is here supposed to be measured from the *upper* side of the inclined plane). If $\frac{\pi}{2} > 2\phi + i$, θ is negative and the applied force will act towards the *under* side.

9. In the first solution of the last example, what is the magnitude of the pressure on the plane?

Ans. Zero. Explain this.

10. If the shaft, (Art. 100), is a square prism of the weight W, and rotates about an axis in its centre, prove that the moment of the friction of the square end varies as the side of the square.

11. If the shaft is composed of two equal circular cylinders placed side by side, and rotates about the line of contact of the two cylinders, show that the moment of the friction of the surface in contact with the horizontal plane $= \frac{32\mu a W}{9\pi}$.

12. What is the least coefficient of friction that will allow of a heavy body's being just kept from sliding down

an inclined plane of given inclination, the body (whose weight is W) being *sustained* by a given horizontal force, P?

Ans. $\dfrac{W \tan i - P}{W + P \tan i}$.

13. It is observed that a body whose weight is known to be W can be just sustained on a rough inclined plane by a horizontal force P, and that it can also be just sustained on the same plane by a force Q up the plane; express the angle of friction in terms of these known forces.

Ans. Angle of friction $= \cos^{-1} \dfrac{PW}{Q\sqrt{P^2 + W^2}}$.

14. It is observed that a force, Q_1, acting up a rough inclined plane will just sustain on it a body of weight W, and that a force, Q_2, acting up the plane will just drag the same body up; find the angle of friction.

Ans. Angle of friction $= \sin^{-1} \dfrac{Q_2 - Q_1}{2\sqrt{W^2 - Q_1 Q_2}}$.

15. A heavy uniform rod rests with its extremities on the interior of a rough vertical circle; find the limiting position of equilibrium.

Ans. If 2α is the angle subtended at the centre by the rod, and λ the angle of friction, the limiting inclination of the rod to the horizon is given by the equation

$$\tan \theta = \frac{\sin 2\lambda}{\cos 2\lambda + \cos 2\alpha}.$$

16. A solid triangular prism is placed, with its axis horizontal, on a rough inclined plane, the inclination of which is gradually increased; determine the nature of the initial motion of the prism.

Ans. If the triangle ABC is the section perpendicular to the axis, and the side AB is in contact with the plane, A being the lower vertex, the initial motion will be one of tumbling if

$$\mu > \frac{b^2 + 3c^2 - a^2}{4\Delta},$$

the sides of the triangle being a, b, c, and its area Δ. If μ is less than this value, the initial motion will be one of slipping.

17. A frustum of a solid right cone is placed with its base on a rough inclined plane, the inclination of which is gradually increased; determine the nature of the initial motion of the body.

Ans. If the radii of the larger and smaller sections are R and r, and h is the height of the frustum, the initial motion will be one of tumbling or slipping according as

$$\mu >< \frac{4R}{h} \cdot \frac{R^2 + Rr + r^2}{R^2 + 2Rr + 3r^2}.$$

18. An elliptic cylinder rests in limiting equilibrium between a rough vertical and an equally rough horizontal plane, the axis of the cylinder being horizontal, and the major axis of the ellipse inclined to the horizon at an angle of 45°. Find the coefficient of friction.

Ans. $\mu = \dfrac{\sqrt{1 + 2e^2 - e^4} - 1}{2 - e^2}$, e being the eccentricity of the ellipse.

CHAPTER VI.

THE PRINCIPLE OF VIRTUAL VELOCITIES.*

101. Virtual Velocity.—*If the point of application of a force be conceived as displaced through an indefinitely small space, the resolved part of the displacement in the direction of the force, is called the Virtual Velocity of the force; and the product of the force into the virtual velocity has been called the virtual moment† of the force.*

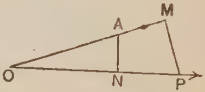

Fig. 50

Thus, let O be the original, and A the new point of application of the force, P, acting in the direction OP, and let AN be drawn perpendicular to it. Then ON is the *virtual velocity* of P, and P · ON is the *virtual moment.* OA is called the *virtual displacement* of the point.

If the projection of the virtual displacement on the line of the force lies on the side of O toward which P acts, as in the figure, the virtual velocity is considered *positive;* but if it lies on the opposite side, *i. e.,* on the action line prolonged through O, it is *negative.* The forces are always regarded as positive; the sign, therefore, of a virtual moment will be the same as that of the virtual velocity.

Cor.—If θ be the angle between the force and the virtual displacement, we have for the virtual moment,

$$P \cdot ON = P \cdot OA \cos\theta = P \cos\theta \cdot OA.$$

---

* The principle of Virtual Velocities was discovered by Galileo, and was very fully developed by Bernouilli and Lagrange.

† Sometimes called "Virtual Work." The name "Virtual Moment" was given by Duhamel.

Now P cos θ is the projection of the force on the direction of the displacement, and is equal to OM, OP being the force and PM being drawn perpendicular to OA. Hence we may also define the virtual moment of a force as *the product of the virtual displacement of its point of application into the projection of the force on the direction of this displacement:* and this definition for some purposes is more convenient than the former.

REMARK.—A force is said to *do work* if it moves the body to which it is applied; and the work done by it is measured by the product of the force into the space through which it moves the body. Generally, the work done by any force during an infinitely small displacement of its point of application is the product of the resolved part of the force in the direction of the displacement into the displacement; and this is the same as the *virtual moment* of the force.

102. Principle of Virtual Velocities.—(1) *The virtual moment of a force is equal to the sum of the virtual moments of its components.*

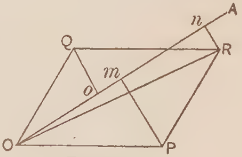

Fig. 51

Let OR represent a force, R, acting at O, and let its components be P and Q, represented by OP and OQ. Let OA be the virtual displacement of O, and let its projections on R, P, and Q, be r, p, and q, respectively. Then the virtual moments of these forces are $R \cdot r$, $P \cdot p$, $Q \cdot q$. Draw Rn, Pm, and Qo, perpendicular to OA. Then On, Om, and Oo ($= mn$), are the projections of R, P, and Q, on the direction of the displacement; and hence (Art. 101, Cor.) we have

$$R \cdot r = \mathrm{OA} \cdot \mathrm{O}n;$$

$$P \cdot p = \mathrm{OA} \cdot \mathrm{O}m;$$

$$Q \cdot q = \mathrm{OA} \cdot mn.$$

Hence
$$P \cdot p + Q \cdot q = OA\,(Om + mn)$$
$$= OA \cdot On = R \cdot r.$$

(See Minchin's Statics, p. 68.)

(2) If there are any number of component forces we may compound them in order, taking any two of them first, and finding the virtual moment of their resultant as above, then finding the virtual moment of the resultant of these two and a third, likewise the virtual moment of the resultant of the first three and a fourth, and so on to the last; or we may use the polygon of forces (Art. 33). The sum of the virtual moments of the forces is equal to the virtual displacement multiplied by the sum of the projections on the displacement of the sides of the polygon which represent the forces (Art. 101, Cor.). But the sum of these projections is equal to the projection of the remaining side of the polygon,* and this side represents the resultant, (Art. 33, Cor. 1). Therefore, *the sum of the virtual moments of any number of concurring forces is equal to the virtual moment of the resultant.*

(3) If the forces are in equilibrium, their resultant is equal to zero; hence, it follows that *when any number of concurring forces are in equilibrium, the sum of their virtual moments* = 0.

This principle is generally known as the *Principle of Virtual Velocities*, and is of great use in the solution of practical problems in Statics.

---

* From the nature of projections (Anal. Geom., Art. 168), it is clear that in any series of points the projection (on a given line) of the line which joins the first and last, is equal to the sum of the projections of the lines which join the points, two and two. Thus, if the sides of a closed polygon, taken in order, be marked with arrows pointing from each vertex to the next one; and if their projections be marked with arrows in the same directions, then, lines measured from left to right being considered positive, and lines from right to left negative, *the sum of the projections of the sides of a closed polygon on any right line is zero.*

103. Nature of the Displacement.—It must be carefully observed that the displacement of the particle on which the forces act is *virtual* and *arbitrary*. The word *virtual* in Statics is used to intimate that the displacements are not really made, but only *supposed*, *i. e.*, they are not *actual* but imagined displacements; but in the motion of a particle treated of in *Kinetics*, the displacement is often taken to be that which the particle *actually* undergoes. In Art. 101, the displacement was limited to an infinitesimal. In some cases, however, a *finite* displacement may be used, and it may be even more convenient to consider a finite displacement. But in very many cases any finite displacement is sufficient to alter the amount or direction of the forces, so as to prevent the *principle of virtual velocities* from being applicable. This difficulty can always be avoided in practice by assuming the displacement to be infinitesimal; and if the virtual displacement is infinitesimal the virtual velocities are all infinitesimal.

104. Equation of Virtual Moments.—Let P_1, P_2, P_3, etc., denote the forces, and δp_1, δp_2, δp_3, etc., the virtual velocities; then the principle of virtual velocities is expressed (Art. 102) by the equation

$$P_1 \cdot \delta p_1 + P_2 \cdot \delta p_2 + P_3 \cdot \delta p_3 + \text{etc.} = 0;$$

or

$$\Sigma P\delta p = 0, \tag{1}$$

which is called the *equation of virtual moments*.*

Sch.—If the virtual displacement is at right angles to the direction of any force, it is clear that δp, the virtual velocity, is equal to zero. Hence, *when the virtual displacement is at right angles to the direction of the force*,

---

* Or *virtual work* (See Art. 101, Rem.). This equation has been made by Lagrange the foundation of his great work on Mechanics, "Mécanique Analytique." (Price's Anal. Mech., Vol. I. p. 142.)

the virtual moment of the force = 0, and *the force will not enter into the equation of virtual moments.*

Such a virtual displacement is always a convenient one to choose when we wish to get rid of some unknown force which acts upon a particle or system.

105. System of Particles Rigidly Connected.—(1) If a particle in equilibrium, under the action of any forces, be constrained to maintain a fixed distance from a given fixed point, the force due to the constraint (if any) is directed towards the fixed point.

Let B be the particle, and A the fixed point. Then it is clear that we may substitute for the string or rigid rod which connects B with A, a smooth circular tube enclosing the particle, with the centre of the tube at A. Now, in order that B may be in equilibrium inside the tube, it is necessary that the resultant of the forces acting upon it should be normal to the tube, *i. e.*, directed towards A.

(2) Let there be any number of particles, m_1, m_2, m_3, etc., each acted on by any forces, P_1, P_2, P_3, etc., and connected with the others by inflexible right lines so that the figure of the system is invariable. Then each particle is acted on by all the *external* forces applied to it, and by all the *internal* forces proceeding from the internal connections of the particle with the other particles of the system. Thus the particle, m, is acted on by P_1, P_2, etc., and by the internal forces which proceed from its connection with m_1, m_2, m_3, etc., and which act along the lines, mm_1, mm_2, etc., by (1) of this Article. Denote the forces along the lines mm_1, mm_2, mm_3, etc., by t_1, t_2, t_3, etc., and their virtual velocities by δt_1, δt_2, δt_3, etc. Now

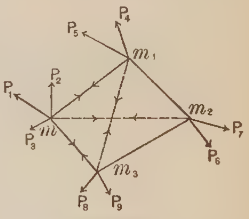

Fig.52

imagine that the system is slightly displaced so as to occupy a new position. Then (1) of Art. 104 gives us for m,

$$P_1\delta p_1 + P_2\delta p_2 + \text{etc.} + t_1\delta t_1 + t_2\delta t_2 + \text{etc.} = 0, \quad (1)$$

for m_1,

$$P_4\delta p_4 + P_5\delta p_5 + \text{etc.} + t_1\delta t_1 + t_2\delta t_2 + \text{etc.} = 0, \quad (2)$$

proceeding in this way as many equations may be formed as there are particles in the system.

Now it is clear that $t_1\delta t_1$, and $t_2\delta t_2$, in (1) have contrary signs from what they have in (2). Thus if the system is moved to the *right* in its displacement, $t_1\delta t_1$, and $t_2\delta t_2$ will be positive in (1) and negative in (2) (Art. 101), and hence, if we add (1) and (2) together, these terms will disappear; in the same way, the virtual moment of the internal force along the line connecting m with any other particle disappears by addition, and the same is true for the internal force between any two particles of the system. Hence, adding together all the equations, the internal forces disappear, and the resulting equation for the whole system is

$$\Sigma P\delta p = 0, \quad (1)$$

and the same result is evidently true whatever be the number of particles forming the system. Hence, *if any number of forces in a system are in equilibrium, the sum of their virtual moments* = 0.

The converse is evidently true, that if the sum of the virtual moments of the forces vanishes for every virtual displacement, the system is in equilibrium.

The following are examples which are solved by the principle of virtual velocities.

EXAMPLES.

1. Determine the condition of equilibrium of a heavy body resting on a smooth inclined plane under the action of given forces.

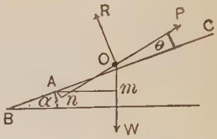

Fig. 53

Let W be the weight of the body sustained on the plane BC by the force, P, making an angle, θ, with the plane. To avoid bringing the unknown reaction, R, into our equation, we make the displacement of its point of application perpendicular to its line of action, (Art. 104, Sch.); hence we conceive O as receiving a virtual displacement, OA, at right angles to R, the magnitude of which in the present case is unlimited. Draw Am and An perpendicular to W and P respectively, Om and On are the virtual velocities of W and P, (Art. 101); and W · Om and P · On are their virtual moments. Hence (1) of Art. 104, gives

$$W \cdot Om - P \cdot On = 0.$$

But $$Om = OA \sin \alpha,$$

and $$On = OA \cos \theta;$$

therefore $$W \sin \alpha - P \cos \theta = 0; \tag{1}$$

which agrees with Ex. 3, Art. 41.

If the force acts parallel to the plane, $\theta = 0$, and (1) becomes

$$P = W \sin \alpha;$$

which agrees with Ex. 1, Art. 41.

2. Suppose the plane in Ex. 1 to be rough, and that the body is on the point of being dragged up the plane, find the condition of equilibrium.

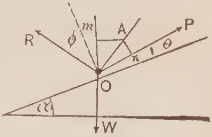

Fig.54

The normal resistance will now be replaced by the total resistance, R, inclined to the normal at an angle $= \phi$, the angle of friction (Art. 95, Cor.). Let the virtual displacement, OA, take place perpendicularly to R, then (1) of Art. 104, gives

$$W \cdot Om - P \cdot On = 0.$$

But $$Om = OA \sin(\alpha + \phi),$$

and $$On = OA \cos(\phi - \theta);$$

therefore $$W \sin(\alpha + \phi) = P \cos(\phi - \theta);$$

which agrees with (3) of Art. 96.

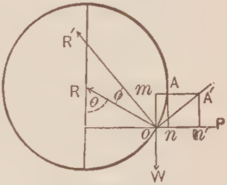

Fig.55

3. Determine the horizontal force which will keep a particle in a given position inside a circular tube, (1) when the tube is smooth and (2) when it is rough.

(1) Let the virtual displacement, OA, be an infinitesimal, $= ds$, along the tube. Then since ds is infinitesimal the virtual velocity of $R = 0$. Then the equation of virtual moments is

$$- W \cdot Om + P \cdot On = 0.$$

But $$Om = ds \cdot \sin\theta,$$

and $$On = ds \cdot \cos\theta;$$

therefore $$W \cdot \sin\theta = P \cdot \cos\theta;$$

or $$P = W \tan\theta.$$

(2) Suppose the force, P, just sustains the particle; the normal resistance must now be replaced by the total resistance, making the angle, ϕ, with the normal at the right of it. Take the virtual displacement, OA′, at right angles to the total resistance (Art. 105, Sch.), and let it be as before, an infinitesimal *ds*. Then (1) of Art. 104, gives

$$-\mathrm{W}\cdot Om+\mathrm{P}\cdot On'=0.$$

But $$Om=ds\cdot\sin(\theta-\phi),$$

and $$On'=ds\cdot\cos(\theta-\phi),$$

therefore $$\mathrm{W}\cdot\sin(\theta-\phi)=\mathrm{P}\cdot\cos(\theta-\phi);$$

or $$\mathrm{P}=\mathrm{W}\cdot\tan(\theta-\phi).$$

Similarly, if the force, P, will *just drag* the particle up the tube we obtain

$$\mathrm{P}=\mathrm{W}\cdot\tan(\theta+\phi).$$

4. Solve by virtual velocities Ex. 6, Art. 62.

Let the displacement be made by diminishing the angle α, which the beam makes with the horizontal plane, by $d\alpha$, the ends of the beam still remaining in contact with the horizontal and vertical planes. Then the virtual velocity of

$$\mathrm{T}=d\cdot 2a\cos\alpha=-2a\sin\alpha\,d\alpha;$$

and that of

$$\mathrm{W}=da\sin\alpha=a\cos\alpha\,d\alpha,$$

and those of the reactions, R and R′, vanish. Then the equation of virtual moments is

$$-\mathrm{T}\,2a \sin \alpha\, d\alpha + \mathrm{W}\, a \cos \alpha\, d\alpha = 0;$$

$$\therefore\quad 2\mathrm{T} \sin \alpha = \mathrm{W} \cos \alpha.$$

5. Solve Ex. 8, Art. 62, by virtual velocities.

Let the displacement be made by increasing the angle θ by $d\theta$, the point, A, remaining in contact with the wall; the virtual displacement of B is at right angles to the direction of the tension, T, and hence the virtual moment of T is zero; the virtual velocity of W is

$$d\,(b \cos \phi - a \cos \theta) = a \sin \theta\, d\theta - b \sin \phi\, d\phi.$$

Then (1) of Art. 104, gives

$$\mathrm{W}\,(a \sin \theta\, d\theta - b \sin \phi\, d\phi) = 0;$$

$$\therefore\quad b \sin \phi\, d\phi = a \sin \theta\, d\theta.$$

But from the geometry of the figure we have

$$b \sin \phi = 2a \sin \theta;$$

$$\therefore\quad b \cos \phi\, d\phi = 2a \cos \theta\, d\theta;$$

$$\therefore\quad 2 \tan \phi = \tan \theta;$$

which, combined with (5) of Ex. 8, Art. 62, gives us the values of $\sin \theta$ and $\cos \phi$; and these in (6) of that Ex. give us the value of x.

6. Solve Ex. 38, Art. 65, by virtual velocities.

Since the bar is to rest in all positions on the curve and the peg, its centre of gravity will neither rise nor fall when the bar receives a displacement, hence its virtual velocity will $= 0$; $\therefore$ etc.

7. In Ex. 4, Art. 42, prove that (1) is the equation of virtual moments.

8. Find the inclination of the beam to the vertical in Ex. 31, Art. 65, by virtual velocities.

9. Deduce, by virtual velocities, (1) the formula for the triangle of forces (see 1 of Art. 32), and (2) the formula for the parallelogram of forces (See 1 of Art. 30).

CHAPTER VII.

MACHINES.

105. Functions of a Machine.—*A machine, Statically, is any instrument by means of which we may change the direction, magnitude, and point of application of a given force; and Kinetically, it is any instrument by means of which we may change the direction and velocity of a given motion.*

In applying the principle of virtual velocities to a system of connected bodies, advantage is gained by choosing the virtual displacements in certain directions (Art. 104, Sch). When we use this principle in the discussion of machines the displacements which we shall choose will be those which the different parts of a machine *actually* undergo when it is employed in doing work, and instead of equations of *virtual* work we shall have equations of *actual* work; and in future the principle of virtual velocities will often be referred to as the *Principle of Work*. (See Minchin's Statics, p. 383.)

Every machine is designed for the purpose of overcoming certain forces which are called *resistances;* and the forces which are applied to the machines to produce this effect are called *moving forces.* When the machine is in motion, every moving force displaces its point of application in its own direction, while the point of application of a resistance is displaced in a direction *opposite* to that of the resistance. Hence, a moving force is one whose elementary work* is *positive*, and a resistance is one whose elementary work is *negative.* The moving force is, for convenience, called the

---

* See Art. 101, Rem.

power; and because the attraction of gravity is the most common form of the force or resistance to be overcome it is usually called the *weight.*

The weight or resistance to be overcome may be the earth's attraction, as in raising a weight; the molecular attractions between the particles of a body as in stamping or cutting a metal, or dividing wood; or friction, as in drawing a heavy body along a rough road. The power may be that of men, or horses, or the steam engine, etc., and may be just sufficient to overcome the resistance, or it may be in excess of what is necessary, or it may be too small. If just sufficient, the machine, if in motion, will remain uniformly so, or if it be at rest it will be on the point of moving, and the power, weight, and friction will be in equilibrium. If the power be in excess, the machine will be set in motion and will continue in accelerated motion. If the power be too small, it will not be able to move the machine; and if it be already in motion it will gradually come to rest.

The general problem with regard to machines is to find the relation between the power and the weight. Sometimes it is most convenient that this relation should be one of equality, *i. e.*, that the power should equal the weight. Generally, however, it is most convenient that the power should be very different from the weight. Thus, if a man has to lift a weight of one ton hanging by a rope, it is clear that he cannot do it unless the mechanical contrivance provided enable him to lift the weight by exercising a pull of very much less, say one cwt. When the power is much smaller than the weight, as it is in this case, which is a very common one, the machine is said to work at a *mechanical advantage.* When, as in some other cases, it is desirable that the power should be greater than the weight, there is said to be a *mechanical disadvantage* of the machine.

107. Mechanical Advantage.—(1) Let P and W be the power and weight, and p and w their virtual velocities respectively; and let friction be omitted. Then from the equation of virtual work (Art. 104), we have

$$Pp - Ww = 0, \quad \text{or} \quad \frac{P}{W} = \frac{w}{p},$$

which shows that the smaller P is in comparison with W, the smaller w will be in comparison with p. But the smaller P is in comparison with W, the greater is the *mechanical advantage*. Hence, the greater the mechanical advantage is the less will be the virtual velocity of the weight in comparison with that of the power. Now, if motion actually takes place the *virtual* velocities become *actual* velocities; and hence we have the principle *what is gained in power is lost in velocity.*

(2) There are no cases in which the weight and power are the only forces to be considered. In every movement of a machine there will always be a certain amount of friction: and this can never be omitted from the equation of virtual work. There are cases, however, as that of a balance on a knife-edge, where the friction is very small; and for these the principle, what is gained in power is lost in velocity, is very approximately true. Where the friction is considerable this is no longer the case.

Let F and f be the resistance of friction and its virtual velocity, then the equation for any machine will take the form

$$Pp - Ww - Ff = 0,$$

which shows us that although P can be made as small as we wish by taking p large enough, yet the mechanical advantage of diminishing P is restricted by the fact that f increases with p; and therefore as P diminishes there is a corresponding increase of the work to be done against friction. Hence if friction be neglected, there is no practical limit to the ratio of P to W; but if the friction be considered, the advantage of diminishing P has a limit, since if Pp remains the same, Ww must decrease as Ff increases; *i. e.*, the work done against friction increases with the complexity of the machine; and thus puts a practical limit to the mechanical advantage which it is possible to obtain by the use of machines.

108. Simple Machines.—The simple machines, sometimes called the *Mechanical Powers*, are generally enumerated as six in number; the *Lever*, the *Wheel and Axle*, the *Inclined Plane*, the *Pulley*, the *Wedge*, and the *Screw*. The *Lever*, the *Inclined Plane*, and the *Pulley*, may be considered as distinct in principle, while the others are combinations of them.

The *efficiency*[*] of a machine is the ratio of the *useful* work it yields to the whole amount of work performed by it. The *useful* work is that which is performed in overcoming *useful* resistances, while *lost* work is that which is spent in overcoming *wasteful* resistances. *Useful* resistances are those which the machine is specially designed to overcome, while the overcoming of *wasteful* resistances is foreign to its purpose. *Friction and rigidity of cords* are *wasteful* resistances while the *weight* of the body to be lifted is the *useful* resistance.

Let W be the work done by the moving forces, W_u the useful and W_l the lost work when the machine is moving uniformly. Then

$$W = W_u + W_l,$$

and if M denote the efficiency of the machine, we have

$$M = \frac{W_u}{W}.$$

In a *perfect* machine, where there is no lost work, the efficiency is unity; but in every machine some of the work is lost in overcoming wasteful resistances, so that the efficiency is always less than unity; and the object of all improvements in a machine is to bring its efficiency as near unity as possible.

The most noticeable of the wasteful resistances are friction and rigidity of cords; and of these we shall consider only the first. The student who wants information on the experimental laws of the rigidity of cords is referred to Weisbach's Mechanics, Vol. I, p. 363.

---

[*] Sometimes called *modulus*.

109. The Lever.—A lever is a rigid bar, straight or curved, movable about a fixed axis, which is called the *fulcrum.* The parts of the lever into which the fulcrum divides it are called the *arms* of the lever. When the arms are in a straight line it is called a *straight lever;* in all other cases it is a *bent lever.*

Levers are divided, for convenience, into three kinds, according to the position of the fulcrum. In the first kind the fulcrum is between the power and the weight; in the second kind the weight acts between the fulcrum and the power; in the third kind the power acts between the fulcrum and the weight. In the last kind the power is always greater than the weight.

A pair of scissors furnishes an example of a pair of levers of the first kind; a pair of nut-crackers of the second kind; and a pair of shears of the third kind.

110. Conditions of Equilibrium of the Lever.—(1) *Without Friction.* Let AB be the lever and C its fulcrum; and let the two forces, P and W, act in the plane of the paper at the points, A and B, in the directions, AP and BW. From C draw CD and CE perpendicular to the directions of P and W. Let α and β denote the angles which the directions of the forces make with the lever. Then, taking moments around C, we have

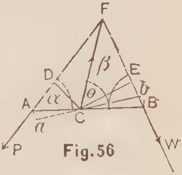

Fig. 56

$$P \cdot \text{CD} = W \cdot \text{CE},$$

or

$$\frac{P}{W} = \frac{\text{perpendicular on direction of } W}{\text{perpendicular on direction of } P}. \qquad (1)$$

That is, the condition of equilibrium requires that *the power and weight should be to each other inversely as the length of their respective arms* (Art. 46).

To find the pressure on the fulcrum, and its direction; let the directions of the pressures, P and W, intersect in F; join C and F; then, since the lever is in equilibrium by the action of the forces, P and W, and the reaction of the fulcrum, the resultant of P and W must be equal and opposite to that reaction, and hence must pass through C and be equal to the pressure on the fulcrum. Denote this resultant by R, the angle which it makes with the lever by θ; and the angle AFB by ω; then we have by (1) of Art. 30

$$R^2 = P^2 + W^2 + 2PW \cos \text{AFB};$$

or

$$R^2 = P^2 + W^2 + 2PW \cos \omega, \tag{2}$$

which gives the pressure, R, on the fulcrum.

To find its direction resolve P, W, and R parallel and perpendicular to the lever, and we have

for parallel forces, $P \cos \alpha - W \cos \beta - R \cos \theta = 0;$

for perpendicular forces, $P \sin \alpha + W \sin \beta - R \sin \theta = 0;$

by transposition and division we get

$$\tan \theta = \frac{P \sin \alpha + W \sin \beta}{P \cos \alpha - W \cos \beta}, \tag{3}$$

which gives the direction of the pressure.

Cor.—When the lever is bent or curved the condition of equilibrium is the same.

Solution by the principle of virtual velocities.

Suppose the lever to be turned round C in the direction of P through the angle $d\theta$, into the position ab; let p and

q be the perpendiculars CD and CE respectively, then the virtual velocity of P will be (Art. 101),

$$\mathrm{A}a \sin \alpha = \mathrm{AC} \cdot d\theta \cdot \sin \alpha = p d\theta.$$

Similarly, the virtual velocity of W is $-q d\theta$.

Hence, by the equation of virtual work we have

$$P \cdot p \cdot d\theta - W \cdot q \cdot d\theta = 0;$$

$$\therefore \quad P \cdot p = W \cdot q. \qquad (4)$$

which is the same as (1).

(2) *With Friction.*—In the above we have supposed friction to be neglected; and if the lever turns round a sharp edge, like the scale beam of a balance, the friction will be exceedingly small. Levers, however, usually consist of flat bars, turning about rounded pins or studs which form the fulcrums, and between the lever and the pin there will of course be friction. To find the friction let r be the radius of the pin round which the lever turns; then the friction on the pin, acting tangentially to the surface of the pin and opposing motion, $= R \sin \phi$ (Art. 99); and the virtual velocity of the point of application of the friction $= r d\theta$; and hence the virtual work of the friction $= R \sin \phi \cdot r d\theta$. Hence the equation of virtual work is

$$P \cdot p d\theta - W \cdot q d\theta - R \sin \phi \, r d\theta = 0.$$

Substituting the value of R from (2), and omitting $d\theta$, we have

$$Pp - Wq = r \sin \phi \sqrt{P^2 + W^2 + 2PW \cos \omega}; \qquad (5)$$

solving this quadratic for P we have

$$P = W\frac{pq + r^2 \cos\omega \sin^2\phi}{p^2 - r^2 \sin^2\phi}$$

$$\pm Wr\sin\phi\frac{\sqrt{p^2 + 2pq\cos\omega + q^2 - r^2\sin^2\phi\sin^2\omega}}{p^2 - r^2\sin^2\phi}, \quad (6)$$

which gives the relation between the power and the weight when friction is considered, the upper or lower sign of $r \sin \phi$ being taken according as P or W is about to preponderate.

Cor.—If the friction is so small that it may be omitted, $r \sin \phi = 0$, and (6) becomes

$$\frac{P}{W} = \frac{q}{p}. \quad (7)$$

111. The Common Balance.—In machines generally the object is to produce motion, not rest; in other words to do work. The statical investigation shows only the limit of force to be applied to put the machine on the *point* of motion, or to give it *uniform* motion. For any work to be done, the force applied must exceed this limit, and the greater the excess, the greater the amount of work done. There is, however, one class of applications of the lever where the object is not to do work, but to produce equilibrium, and which are therefore specially adapted for treatment by statics. This is the class of measuring machines, where the object is not to overcome a particular resistance, but to measure its amount. The testing machine is a good example, measuring the pull which a bar of any material will sustain before breaking. The common balance and steelyard for weighing, are familiar examples.

The common balance is an instrument for weighing; it is a lever of the first kind, with two equal arms, with a scale-pan suspended from each extremity, the fulcrum being vertically above the centre of gravity of the beam when the latter is horizontal, and therefore vertically above the centre of gravity of the system formed by the beam, the scale-pans, and the weights of the scale-pans. The substance to be weighed is placed in one scale-pan, and weights of known magnitude are placed in the other till the beam remains in equilibrium in a perfectly horizontal position, in which case the weight of the substance is indicated by the weights which balance it. If these weights differ by ever so little the horizontality of the beam will be disturbed, and after oscillating for a short time, in consequence of the fulcrum being placed *above* the centre of gravity of the system, it will rest in a position inclined to the horizon at an angle, the extent of which is a measure of the sensibility of the balance.

The preceding explanation represents the balance in its simplest form; in practice there are many modifications and contrivances introduced. Much skill has been expended upon the construction of balances, and great delicacy has been obtained. Thus, the beam should be suspended by means of a knife-edge, *i. e.*, a projecting metallic edge transverse to its length, which rests upon a plate of agate or other hard substance. The chains which support the scale-pans should be suspended from the extremities of the beam in the same manner. The point of support of the beam (fulcrum) should be at equal distances from the points of suspension of the scales; and when the balance is not loaded the beam should be horizontal. We can ascertain if these conditions are satisfied by observing whether there is still equilibrium when the substance is transferred to the scale which the weight originally occupied and the weight to that which the substance originally occupied.

The chief requisites of a good balance are:

(1) When equal weights are placed in the scale-pans the beam should be perfectly horizontal.

(2) The balance should possess great *sensibility*; *i. e.*, if two weights which are very nearly equal be placed in the scale-pans, the beam should vary *sensibly* from its horizontal position.

(3) When the balance is disturbed it should readily return to its state of rest, or it should have *stability*.

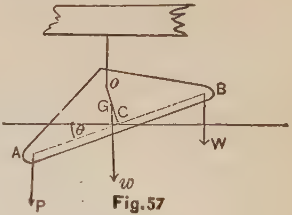

Fig.57

112. To Determine the Chief Requisites of a Good Balance.—Let P and W be the weights in the scale-pans; O the fulcrum; h its distance from the straight line, AB, which joins the points of attachment of the scale-pans to the beam; G the centre of gravity of the beam; and let AB be at right angles to OC, the line joining the fulcrum to the centre of gravity of the beam. Let $AC = CB = a$; $OG = k$; $w =$ the weight of the beam; and $\theta =$ the angle which the beam makes with the horizon when there is equilibrium.

Now the perpendicular from O

on the direction of $P = a \cos \theta - h \sin \theta$;
" " " $W = a \cos \theta + h \sin \theta$;
" " " $w = k \sin \theta$;

therefore taking moments round O we have

$$P(a \cos \theta - h \sin \theta) - W(a \cos \theta + h \sin \theta) - wk \sin \theta = 0;$$

$$\therefore \quad \tan \theta = \frac{(P - W)a}{(P + W)h + wk}. \tag{1}$$

This equation determines the position of equilibrium. The *first* requisite—the horizontality of the beam when P and W are equal—is satisfied by making the arms equal.

The *second* requisite [(2) of Art. 111], requires that, for a given value of $P - W$, the inclination of the beam to the horizon must be as great as possible, and therefore the sensibility is greater the greater $\tan \theta$ is for a given value of $P - W$; and for a given value of $\tan \theta$ the sensibility is greater the smaller the value of $P - W$ is; hence the sensibility may be measured by $\frac{\tan\theta}{P - W}$, which requires that

$$(P + W)\frac{h}{a} + w\frac{k}{a}$$

be as small as possible. Therefore a must be large, and w, h, and k must be small; *i. e.*, the arms must be long, the beam light, and the distances of the fulcrum from the beam and from the centre of gravity of the beam must be small.

The *third* requisite, its stability, is greater the greater the moment of the forces which tend to restore the beam to its former position of rest when it is disturbed. If $P = W$ this moment is

$$[(P + W)\,h + wk]\sin\theta,$$

which should be made as large as possible to secure the third requisite.

This condition is, to some extent, at variance with the second requisite. They may both be satisfied, however, by making $(P + W)\,h + wk$ large and a large also; *i. e.*, by increasing the distances of the fulcrum from the beam and from the centre of gravity of the beam, and by lengthening the arms. (See Todhunter's Statics, p. 180, also Pratt's Mechanics, p. 78.)

The comparative importance of these qualities of *sensibility* and *stability* in a balance will depend upon the use for which it is intended; for weighing heavy weights, *stability* is of more importance; for use in a chemical laboratory the balance must possess great *sensibility;* and instruments have been constructed which indicate a variation of weight less than a *millionth* part of the whole. In a balance of great delicacy the fulcrum is made as thin as possible; it is generally a *knife-edge* of hardened steel or agate, resting on a polished agate plate, which is supported on a strong vertical pillar of brass.

113. The Steelyard.—This is a kind of balance in which the arms are unequal in length, the longer one being graduated, along which a *poise* may be moved in order to balance different weights which are placed in a scale-pan on the short-arm. While the moment of the substance weighed is changed by increasing or diminishing its quantity, its arm remaining constant, that of the poise is changed by altering its arm, the weight of the poise remaining the same.

114. To Graduate the Common Steelyard.—(1) *When the point of suspension is coincident with the centre of gravity.*

Let AF be the beam of the steelyard suspended about an axis passing through its centre of gravity, C; on the arm, CF, place a movable weight, P; then if a weight, W, equal to P, is suspended from A, the beam will balance when P on the long arm is at a distance from C equal to AC. If W equals twice the weight of P, the beam will balance when the distance of P from C is twice AC; and so on in any proportion. Hence if W is successively 1 lb., 2 lbs., 3 lbs., etc., the distances of the notches, 1, 2, 3, 4, etc., where P is placed, are as 1, 2, 3, etc., *i. e.*, the arm CF is divided into *equal* divisions, beginning at the fulcrum, C, as the zero point.

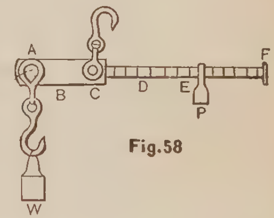

Fig.58

(2) *When the point of suspension is not coincident with the centre of gravity.*

Let C be the fulcrum, W the substance to be weighed, hanging at the extremity, A, and P the movable weight. Suppose that when W is removed, the weight, P, placed at B will balance the long arm, CF, and keep the steelyard in a horizontal position; then the moment of the instrument itself, about C, is on the side, CF, and is equal to $P \cdot \mathrm{CB}$. Hence, if W hangs from A, and P from any point E, then for equilibrium we must have

$$P \cdot \mathrm{CE} + P \cdot \mathrm{BC} = W \cdot \mathrm{AC};$$

or

$$P \cdot \mathrm{BE} = W \cdot \mathrm{AC};$$

$$\therefore \quad \mathrm{BE} = \frac{W}{P} \cdot \mathrm{AC}.$$

If we make W successively equal to P, $2P$, $3P$, etc., then the values of BE will be AC, 2AC, 3AC, etc., and these distances must be measured off, commencing at B for the zero point, and the points so determined marked 1, 2, 3, 4, etc. Such a steelyard cannot weigh below a certain limit, corresponding to the first notch, 1.

To find the length of the divisions on the beam, divide BE, the distance of the poise from the zero point, by the weight, W, which P balances when at the point E. The steelyard often has *two* fulcrums, one for small and the other for large weights.

EXAMPLES.

1. What force must be applied at one end of a lever 12 ins. long to raise a weight of 30 lbs. hanging 4 ins. from the fulcrum which is at the other end, and what is the pressure on the fulcrum? *Ans.* 10 lbs. : 20 lbs.

2. A lever weighs 3 lbs., and its weight acts at its middle point; the ratio of its arms is 1 : 3. If a weight of 48 lbs. be hung from the end of the shorter arm, what weight must be suspended from the other end to prevent motion? *Ans.* 15 lbs.

3. The arms of a *bent* lever are 3 ft. and 5 ft. and inclined to each other at an angle $\theta = 150°$. To the short arm a weight of 7 lbs. is applied and to the long arm a weight of [illegible] lbs. is applied. Required the inclination of each arm to the horizon when there is equilibrium.

Ans. The short arm is inclined at an angle of 18° 22′ *above* the horizon, and the long arm is inclined at an angle of 48° 22′ *below* the horizon.

115. The Wheel and Axle.—This machine consists of a wheel, *a*, rigidly connected with a horizontal cylinder, *b*, movable round two trunnions (Art. 99), one of which is shown at *c*. The power, *P*, is applied at the circumference of the wheel, sometimes by a cord coiled round the wheel, sometimes by handspikes as in the *capstan*, or by handles as in the *windlass*; the weight, *W*, hangs at the end of a cord fastened to the axle and coiled round it.

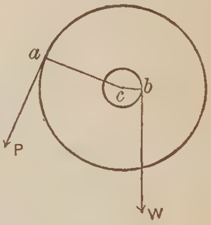

Fig. 59

116. Conditions of Equilibrium of the Wheel and Axle.—(1) Let a and b be the radii of the wheel and axle respectively; P and W the power and weight, supposed to act by strings at the circumference of the wheel and axle perpendicular to the radii a and b. Then either by the principle of virtual velocities or by the principle of moments we have

$$Pa = Wb,$$

or

$$\frac{P}{W} = \frac{\text{radius of axle}}{\text{radius of wheel}}. \qquad (1)$$

It is evident that, by increasing the radius of the wheel or by diminishing the radius of the axle, any amount of mechanical advantage may be gained. It will also be seen that this machine is only a modification of the lever; the peculiar advantage of the wheel and axle being that an endless series of levers are brought into play. In this respect, then, it surpasses the common lever in mechanical advantage.

In the above we have supposed friction to be neglected, or, what amounts to the same thing, have assumed that the trunnion is indefinitely small. In practice, of course, the trunnion has a certain radius, r, and a certain coefficient of friction. Calling R the resultant of P and W, and taking into account the friction on the trunnion we have for the relation between P and W

$$Pa = Wb + r \sin \phi \sqrt{P^2 + W^2 + 2PW \cos \omega}. \qquad (2)$$

ω being the angle between the directions of P and W exactly as in Art. 110.

(2) **Differential Wheel and Axle.**—By diminishing b, the radius of the axle, the strength of the machine is diminished; to avoid this disadvantage a *differential wheel and axle* is sometimes employed. In this instrument the axle consists of two cylinders of radii b and b'; the rope is wound round the former in one direction, and after passing under a movable pulley to which the weight is attached, is wound round the latter in the opposite direction, so that as the power, P, which is applied as before, tangentially to the wheel of radius, a, moves in its own direction, the rope at b winds up while the rope at b' unwinds.

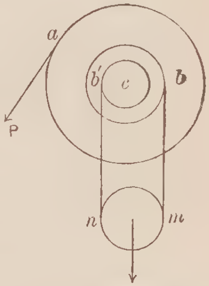

Fig. 60

For the equilibrium of the forces (whether at rest or in uniform motion), the tensions of the rope in bm and $b'n$

are each equal to $\frac{1}{2}W$. Hence, taking moments round the centre of the trunnion, c, we have

$$Pa + \tfrac{1}{2}Wb' - \tfrac{1}{2}Wb = 0;$$

$$\therefore \quad Pa = \tfrac{1}{2}W(b - b'), \tag{3}$$

hence by making the difference, $b - b'$, small, the power can be made as small as we please to lift a given weight. Let the wheel turn through the angle $\delta\theta$; the point of application of P will describe a space $= a\delta\theta$, and the weight will be lifted through a space $= \frac{1}{2}(b - b')\,\delta\theta$, which latter will be very small if $b - b'$ is very small. Therefore, since the amount of *work* to be done to raise the weight to any given height, is constant, economy of power is accomplished by a loss in the time of performing the work.

117. Toothed Wheels.—*Toothed or cogged wheels* are wheels provided on the circumferences with projections called teeth or cogs which interlock, as shown in the figure, and which are therefore capable of transmitting force, so that if one of the wheels be turned round by any means, the other will be turned round also.

When the teeth are on the *sides* of the wheel instead of the circumference, they are called *crown* wheels. When the axes of two wheels are neither perpendicular nor parallel to each other, the wheels take the form of frustums of cones, and are called *beveled wheels.* When there is a pair of toothed wheels on each axle with the teeth of the large one on one axle fitting between the teeth

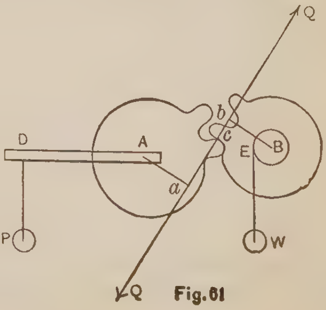

Fig. 61

of the small one on the next axle, the larger wheel of each pair is called the wheel, and the smaller is called the *pinion.* By means of a combination of toothed wheels of this kind called a train of wheels, motion may be transferred from one point to another and work done, each wheel driving the next one in the series. The discussion of this kind of machinery possesses great geometric elegance ; but it would be out of place in this work. We shall give only a slight sketch of the simplest case, that in which the axes of the wheels are all parallel. For the investigation of the proper forms of teeth in order that the wheels when made shall run truly one upon another the student is referred to other works.*

118. To Find the Relation of the Power and Weight in Toothed Wheels.—Let A and B be the fixed centres of the toothed wheels on the circumferences of which the teeth are arranged ; QCQ a normal to the surfaces of two teeth at their point of contact, C. Suppose an axle is fixed on the wheel, B, and the weight, W, suspended from it at E by a cord ; also, suppose the power, P, acts at D with an arm DA ; draw Aa and Bb perpendicular to QCQ. Let Q be the mutual pressure of one tooth upon another at C ; this pressure will be in the direction of the normal QCQ. Now since the wheel, A, is in equilibrium about the fixed axis, A, under the action of the forces, P and Q, we have

$$P \cdot \mathrm{AD} = Q \cdot \mathrm{A}a; \tag{1}$$

and since the wheel, B, is in equilibrium about the fixed axis, B, under the action of the forces, Q and W, we have

$$W \cdot \mathrm{BE} = Q \cdot \mathrm{B}b. \tag{2}$$

---

* See Goodeve's *Elements of Mechanism* : Rankine's *Applied Mechanics* ; Moseley's *Engineering* ; Willis's *Principles of Mechanism* ; Collignon's *Statique* ; and a *Paper of* Mr. Airy's *in the Camb. Phil. Trans.*, Vol. II, p. 277.

Dividing (1) by (2) we have

$$\frac{P \cdot \mathrm{AD}}{W \cdot \mathrm{BE}} = \frac{\mathrm{A}a}{\mathrm{B}b};$$

or

$$\frac{\text{moment of } P}{\text{moment of } W} = \frac{\mathrm{A}a}{\mathrm{B}b}.$$

If the direction of the normal, QCQ, at the point of contact, C, changes as the action passes from one tooth to the succeeding, the relation of P to W becomes variable. But, if the teeth are of such form that the normal at their point of contact shall always be tangent to both wheels, the lines Aa and Bb will become radii, and their ratio constant. And since the number of teeth in the two wheels is proportional to their radii, we have

$$\frac{\text{moment of } P}{\text{moment of } W} = \frac{\text{number of teeth on the wheel } P}{\text{number of teeth on the wheel } W}. \quad (3)$$

119. Relation of Power to Weight in a Train of n Wheels.—Let R_1, R_2, R_3, etc., be the radii of the successive wheels in such a train; r_1, r_2, r_3, etc., the radii of the corresponding pinions; and let P, P_1, P_2, P_3, . . . W, be the powers applied to the circumferences of the successive wheels and pinions. Then the first wheel is in equilibrium about its axis under the action of the forces P and P_1, since the power applied to the circumference of the second wheel is equal to the reaction on the first pinion, therefore

$$P \times R_1 = P_1 \times r_1.$$

Similarly

$$P_1 \times R_2 = P_2 \times r_2;$$

$$P_2 \times R_3 = P_3 \times r_3;$$

$$\text{etc.} = \text{etc.};$$

$$P_{n-1} \times R_n = W \times r_n.$$

Multiplying these equations together and omitting common factors, we have

$$\frac{P}{W} = \frac{r_1 \times r_2 \times r_3 \times \dots .}{R_1 \times R_2 \times R_3 \times \dots .}. \qquad (1)$$

It will be observed, in toothed gearing, that the smaller the radius of the pinion as compared with the wheel, the greater will be the mechanical advantage. There is, however, a practical limit to the size that can be given to the pinion, because the teeth must be large enough for strength, and must not be too few in number. Six is generally the least number admissible for the teeth of a pinion. Equation (1) shows that by a train consisting of a very few pairs of wheels and pinions there is an enormous mechanical advantage. Thus, if there are three pairs, and the ratio of each wheel to the pinion is 10 to 1, then P is only one thousandth part of W; but on the other hand, W will only make one turn where P makes one thousand. Such trains of wheels are very useful in machinery such as hand cranes, where it is not essential to obtain a quick motion, and where the power available is very small in comparison to the weight. (See Browne's Mechanics, p. 109.)

EXAMPLES.

1. What is the diameter of a wheel if a power of 3 lbs. is just able to move a weight of 12 lbs. that hangs from the axle, the radius of the axle being 2 ins.? *Ans.* 16 ins.

2. If a weight of 20 lbs. be supported on a wheel and axle by a force of 4 lbs., and the radius of the axle is $\frac{2}{3}$ in., find the radius of the wheel. *Ans.* $3\frac{1}{3}$ ins.

3. A capstan is worked by a man pushing at the end of a pole. He exerts a force of 50 lbs., and walks 10 ft. round for every 2 ft. of rope pulled in. What is the resistance overcome? *Ans.* 250 lbs.

4. An axle whose diameter is 10 ins., has on it two wheels the diameters of which are 2 ft. and $2\frac{1}{2}$ ft. respectively. Find the weight that would be supported on the axle by weights of 25 lbs. and 24 lbs. on the smaller and larger wheels respectively. *Ans.* 132 lbs.

120. The Inclined Plane.—This has already been partly considered (Art. 96, etc.). Let the power, P, whose direction makes an angle, θ, with a rough inclined plane, be employed to drag a weight, W, up the plane. Then if ϕ is the angle of friction and i the inclination of the plane, we have from (3) of Art. 96,

$$P = W\frac{\sin(i + \phi)}{\cos(\phi - \theta)}. \tag{1}$$

If P acts along the plane, $\theta = 0$, and (1) becomes

$$P = W\frac{\sin(i + \phi)}{\cos\phi}. \tag{2}$$

If P acts horizontally, $\theta = -i$, and (1) becomes

$$P = W\tan(i + \phi). \tag{3}$$

Cor.—If we suppose the friction $= 0$, (1), (2), and (3) become respectively

$$P = W\frac{\sin i}{\cos\theta}, \tag{4}$$

$$P = W\sin i, \tag{5}$$

$$P = W\tan i. \tag{6}$$

Sch.—It follows from (4), (5), and (6) that the smaller the inclination* of the plane to the horizon, the greater will be the mechanical advantage. If we take in friction there is an exception to this rule when $i > \frac{\pi}{2} - \phi$. The gradients on railways are the most common examples of the use of the inclined plane; these are always made as low as is convenient in order to enable the engine to lift the heaviest possible train.

121. The Pulley.—The *pulley* consists of a *grooved wheel*, capable of revolving freely about an axis, fixed into a framework, called the *block*. A cord passes over a portion of the circumference of the wheel in the groove. When the axis of the pulley is fixed, the pulley is called a *fixed* pulley, and its only effect is to change the direction of the force exerted by the cord; but where the pulley can ascend and descend it is called a *movable* pulley, and a mechanical advantage may be gained. Combinations of pulleys may be made in endless variety; we shall consider only the simple movable pulley and three of the more ordinary combinations. No account will be here taken of the weight of the pulleys or of the cord, or of friction and stiffness of cords. The weight of a set of pulleys is generally small in comparison with the loads which they lift; and the friction is small. The use of the pulley is to diminish the effects of friction which it does by transferring the friction between the cord and circumference of the wheel to the axis and its supports, which may be highly polished or lubricated. The mechanical principle involved in all calculations with respect to the pulley is the constancy of the force of tension in all parts of the same string (Art. 40).

---

[*] To find the inclination of the plane for a maximum value of P when it acts parallel to the plane we put the derivative of P with respect to $i = 0$, and get $\frac{dP}{di} = W\frac{\cos(i+\phi)}{\cos\phi} = 0$, $\therefore i = \frac{\pi}{2} - \phi$. Hence while the inclination of the plane is diminishing from $\frac{\pi}{2}$ to $\frac{\pi}{2} - \phi$, mechanical advantage is diminishing.

122. The Simple Movable Pulley.—Let O be the centre of the pulley which is supported by a cord passing under it with one end attached to a beam at A and the other end stretched by the force P.

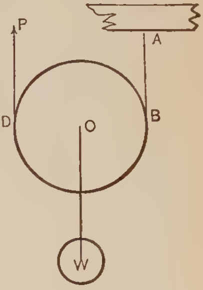

Fig. 62

Now since the tension of the string, ABDP, is the same throughout, and the weight, W, is supported by the two strings at B and D, in each of which the tension is P, we have

$$2P = W; \quad \therefore \quad \frac{P}{W} = \frac{1}{2}.$$

The same result follows by the principle of virtual velocities. Suppose the pulley and the weight, W, to rise any distance. Then it is clear that both halves of the string must be shortened by the same distance, and hence P must rise double the distance; and therefore the equation of virtual work gives

$$2P = W; \quad \therefore \quad \frac{P}{W} = \frac{1}{2}.$$

The mechanical advantage with a single movable pulley is 2.

123. First System of Pulleys, in which the same cord passes round all the Pulleys.—In this system there are two blocks, A and B, the upper of which is fixed and the lower movable, and each containing a number of pulleys, each pulley being movable round the axis of the block in which it is. A single cord is attached to the lower block and passes alternately round the pulleys in the upper and lower blocks, the portions of the cord between successive pulleys being parallel. The portion

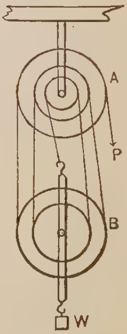

Fig. 63

of cord proceeding from one pulley to the next is called a *ply*; the portion at which the power, P, is applied is called the *tackle-fall*.

Since the cord passes round all the pulleys its tension is the same throughout and equal to P. Then if n be the number of plies at the lower block, nP will be the resultant upward tension of the cords at the lower block, which must equal W;

$$\therefore \quad nP = W,$$

or

$$\frac{P}{W} = \frac{1}{n}.$$

This result follows also by the principle of virtual velocities. Let p denote the length of the tackle-fall and x the common length of the plies; then since the length of the cord is constant, we have

$$p + nx = \text{constant};$$

$$\therefore \quad dp + ndx = 0.$$

But the equation of virtual work is

$$Pdp + Wdx = 0;$$

$$\therefore \quad P = \frac{W}{n}, \quad \text{or} \quad \frac{P}{W} = \frac{1}{n}.$$

This system is most commonly used on account of its superior portability and is the only one of practical importance. The several pulleys are usually mounted on a common axis, as in the figure, the cord being inclined slightly *aside* to pass from one pair of pulleys to the next.

This forms what is called a set of *Blocks and Falls*. It is very commonly used on shipboard and wherever weights have to be lifted at irregular times and places. The weight of the lower set of pulleys in this case merely forms part of the gross weight W.

The friction on the spindle of any particular pulley is proportional to the total pressure on the pulley, which is clearly $2P$. Hence, if μ is the coefficient of friction, the resistance of friction on any pulley $= 2P\mu$; and the amount of its displacement, when W is raised, will be to the displacement of W in the ratio of the radius of the spindle to that of the pulley.

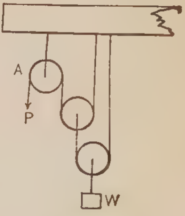

Fig. 64

124. Second System of Pulleys, in which each Pulley hangs from a fixed block by a separate String.—Let A be the fixed pulley, n the number of movable pulleys; each cord has one end attached to a fixed point in the beam, and all except the last have the other end attached to a movable pulley, the portions not in contact with any pulley being all parallel.

Then the tension of the cord passing under the first (lowest) pulley $= \frac{W}{2}$ (Art. 122); the tension of the cord passing under the second pulley $= \frac{W}{2^2}$, and so on; and the tension of the cord passing under the nth pulley $= \frac{W}{2^n}$, which must equal the power, P;

$$\therefore \quad \frac{P}{W} = \frac{1}{2^n}. \tag{1}$$

The same result follows by the principle of work. Suppose the first pulley and the weight W to rise any distance, x; then it is clear that both portions of the cord passing round this pulley will be shortened by the same distance, and hence the second pulley must rise double this distance or $2x$, and the third pulley must rise double the distance of the second or 2^2x, and so on; and the nth pulley must rise $2^{n-1}x$ and P must descend 2^nx; therefore the work of P

is $P2^nx$, and the work to be done on W is $W\cdot x$. Hence the equation of work gives

$$P\cdot 2^n x = Wx, \quad \therefore \quad \frac{P}{W} = \frac{1}{2^n}.$$

125. Third System of Pulleys, in which each cord is attached to the weight.—In this system one end of each cord is attached to the bar from which the weight hangs, and the other supports a pulley, the cords being all parallel, and the number of movable pulleys one less than the number of cords.

Let n be the number of cords; then the tension of the cord to which P is attached is P; the tension of the second cord is $2P$ (Art. 122); that of the next 2^2P, and so on; and the tension of the nth cord is $2^{n-1}P$. Then the sum of all the tensions of the cords attached to the weight must equal W. Hence

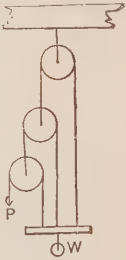

Fig: 65

$$P + 2P + 2^2P + \ldots . 2^{n-1}P = (2^n - 1)P = W;$$

$$\therefore \quad \frac{P}{W} = \frac{1}{2^n - 1}.$$

In this system the weights of the movable pulleys assist P; in the two former systems they act against it.

EXAMPLES.

1. What force is necessary to raise a weight of 480 lbs. by an arrangement of six pulleys in which the same string passes round each pulley? *Ans.* 80 lbs.

2. Find the power which will support a weight of 800 lbs. with three movable pulleys, arranged as in the second system. *Ans.* 100 lbs.

3. If there be equilibrium between P and W with three pulleys in the third system, what additional weight can be raised if 2 lbs. be added to P? *Ans.* 14 lbs.

126. The Wedge.—The wedge is a triangular prism, usually isosceles, and is used for separating bodies or parts of the same body by introducing its edge between them and then thrusting the wedge forward. This is effected by the blow of a hammer or other such means, which produces a violent pressure, for a short time, in a direction perpendicular to the back of the wedge, and the resistance to be overcome consists of friction and a reaction due to the molecular attractions of the particles of the body which are being separated. This reaction will be in a direction perpendicular to the inclined surface of the wedge.

127. The Mechanical Advantage of the Wedge.—Let ACB represent a section of the wedge perpendicular to its inclined faces, the wedge having been driven into the material a distance equal to DC by a force, P, acting in the direction DC. Draw DE, DF, perpendicular to AC, BC, and let R denote the reactions along ED and FD; then μR will be the friction acting at E and F in the directions EA and FB. Let the angle of the wedge or ACB $= 2\alpha$.

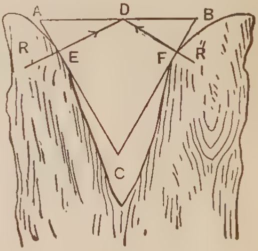

Fig. 66

Resolve the forces which act on the wedge in directions perpendicular and parallel to the back of the wedge, then we have for perpendicular forces

$$P = 2R \sin \alpha + 2\mu R \cos \alpha. \qquad (1)$$

This equation may also be obtained from the principle of work as follows: If the wedge has been driven into the material a distance equal to DC by a force, P, acting in the direction DC, then the work done by P is $P \times \mathrm{DC}$ (Art. 101, Rem.); and since the points E and F were originally together, the work done against the resistance R is $R \times \mathrm{DE} + R \times \mathrm{DF} = 2R \times \mathrm{DE}$; and the work done against friction is $2\mu R \times \mathrm{EC}$. Hence the equation of work is

$$P \times \mathrm{DC} = 2R \times \mathrm{DE} + 2\mu R \times \mathrm{EC}, \qquad (2)$$

which reduces to (1) by substituting $\sin \alpha$ and $\cos \alpha$ for $\frac{\mathrm{DE}}{\mathrm{DC}}$ and $\frac{\mathrm{EC}}{\mathrm{DC}}$.

Cor.—If friction be neglected, (2) becomes

$$\frac{P}{R} = \frac{2\mathrm{DE}}{\mathrm{DC}} = \frac{\mathrm{AB}}{\mathrm{AC}},$$

that is $$\frac{P}{R} = \frac{\text{back of the wedge}}{\text{length of one of the equal sides}}.$$

It follows that the narrower the back of the wedge, the greater will be the mechanical advantage. Knives, chisels, and many other implements are examples of the wedge.

In the action of the wedge a great part of the power is employed in cleaving the material into which it is driven. The force required to effect this is so great that instead of applying a continuous pushing force perpendicular to the back of the wedge, it is driven by a series of blows. Between the blows there is a powerful reaction, R, acting to push the wedge back again out of the cleft, and this is resisted by the friction which now acts in the directions EC and FC. Hence when the wedge is on the point of starting back, between the blows, the equation of equilibrium will be from (1)

$$2R \sin \alpha - 2\mu R \cos \alpha = 0;$$

$$\therefore \quad \alpha = \tan^{-1} \mu.$$

And the wedge will fly back or not according as $\alpha >$ or $< \tan^{-1}\mu$. (See Browne's Mechanics, p. 117. Also Magnus's Mechanics, p. 157.)

128. The Screw.—The screw consists of a right circular cylinder, on the convex surface of which there is traced a uniform projecting thread, *abcd* . . . . inclined at a constant angle to straight lines parallel to the axis of the cylinder. The path of the thread may be traced by the edge AC of an inclined plane, ABC, wrapped round the cylinder; the base of the plane corresponding with the circumference of the cylinder, and the height of the plane with the distance between the threads which is called the *pitch* of the screw. The threads may be rectangular or triangular in section. The cylinder fits into a block, on the inner surface of which is cut a groove which is the exact counterpart of the thread. The block in which the groove is cut is often called the *nut*. The power is generally applied at the end of a lever fixed to the centre of the cylinder, or fixed to the nut. It is evident that a screw never requires any pressure in the direction of its axis, but must be made to revolve only: and this can be done by a force acting at right angles to the extremities of its diameter, or its diameter produced.

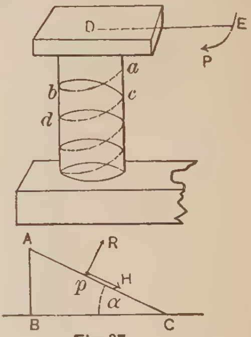

Fig. 67

129. The Relation between the Power and the Weight in the Screw.—Suppose the power, P. to act in a plane perpendicular to the axis of the cylinder and at the end of an arm, DE $= a$, and suppose the screw to have made one revolution, the power, P, will have moved through the circumference of which a, is the radius, and the work done by P will be $P \times 2\pi a$. During the same time the screw will have moved in the direction of its axis through the distance, AB $= 2\pi r \tan \alpha$, r being the radius of the cylinder, and α the angle which the thread of the screw makes with its base. Then as this is the direction in which the resistance is encountered, the work done against the resistance, W, is $W 2\pi r \tan \alpha$. Hence if no work is lost the equation of work will be

$$P \times 2\pi a = W \times 2\pi r \tan \alpha. \tag{1}$$

That is *the power is to the weight as the pitch of the screw is to the circumference described by the power.*

If there is friction between the thread and the groove, let R be the normal pressure at any point, p, of the thread, and μR the friction at this point, then the work done against the friction in one revolution is $\mu \Sigma R \, 2\pi r \sec \alpha$, ΣR denoting the sum of the normal reactions at all points of the thread. Hence the equation of work is

$$P \, 2\pi a = W \, 2\pi r \tan \alpha + \mu \, 2\pi r \sec \alpha \Sigma R. \tag{2}$$

But, for the equilibrium of the screw, resolving parallel to the axis, we have

$$W = \Sigma \, (R \cos \alpha - \mu R \sin \alpha),$$

therefore $$\Sigma R = \frac{W}{\cos \alpha - \mu \sin \alpha};$$

which in (2) gives

$$Pa = Wr \tan \alpha + \frac{\mu r \sec \alpha W}{\cos \alpha - \mu \sin \alpha};$$

or $$Pa = Wr \tan (\alpha + \phi), \tag{3}$$

ϕ being the angle of friction.

129a. Prony's Differential Screw.—If h denote the pitch of a screw (1) becomes

$$2P\pi a = Wh,$$

which expresses the relation between P and W, when friction is neglected. Therefore the mechanical advantage is gained by making the pitch very small. In some cases, however, it is desirable that the screw should work at fair speed, as in ordinary bolts and nuts, and then the pitch must not be too small. In cases where the screw is used specially to obtain pressure, as in screw-presses for cotton, etc., we do not care for speed, but only for pressure. But in practice it is impossible to get the pitch very small from the fact that if the angle of inclination is very flat, the threads run so near each other as to be too weak, in which case the screw is apt to "strip its thread," that is, to tear bodily out of the hole, leaving the thread behind.

Where very great pressure is required a differential nut-hole is resorted to. Let the screw work in two blocks, A and B, the first of which is fixed and the second movable along a fixed groove, n; and let h be the pitch of the thread which works in the block, A, and h' the pitch of the thread which works in the block B. Then one revolution of the screw impresses two opposite motions on the block, B, one equal to h in the direction in which the screw advances, and the other equal to h' in the opposite direction. If then the block, B, is connected with the resistance W, we have by the principle of work

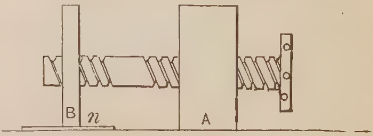

Fig. 68

$$2P\pi a = W(h - h');$$

and the requisite power will be diminished by diminishing

$h - h'$. By means of this screw a comparatively small pressure may be made to yield a pressure enormously greater in magnitude.

EXAMPLES.

1. A lever 10 ins. long, the weight of which is 4 lbs., and acts at its middle point, balances about a certain point when a weight of 6 lbs. is hung from one end; find the point. *Ans.* 2 ins. from the end where the weight is.

2. A lever weighing 8 lbs. balances at a point 3 ins. from one end and 9 ins. from the other. Will it continue to balance about that point if equal weights be suspended from the extremities?

3. A beam whose length is 12 ft. balances at a point 2 ft from one end; but if a weight of 100 lbs. be hung from the other end it balances at a point 2 ft. from that end; find the weight of the beam. *Ans.* 25 lbs.

4. A lever 7 feet long is supported in a horizontal position by props placed at its extremities: find where a weight of 28 lbs. must be placed so that the pressure on one of the props may be 8 lbs. *Ans.* Two feet from the end.

5. Two weights of 12 lbs. and 8 lbs. respectively at the ends of a horizontal lever 10 feet long balance: find how far the fulcrum ought to be moved for the weights to balance when each is increased by 2 lbs. *Ans.* Two inches.

6. A lever is in equilibrium under the action of the forces P and Q, and is also in equilibrium when P is trebled and Q is increased by 6 lbs.: find the magnitude of Q.
Ans. 3 lbs.

7. In a lever of the first kind, let the power be 217 lbs., the weight 725 lbs., and the angle between them 126°. Find the pressure on the fulcrum. *Ans.* 622.7 lbs.

8. If the power and weight in a straight lever of the first kind be 17 lbs. and 32 lbs., and make with each other an angle of 79°; find the pressure on the fulcrum.

Ans. 39 lbs.

9. The length of the beam of a false balance is 3 ft. 9 ins. A body placed in one scale balances a weight of 9 lbs. in the other; but when placed in the other scale it balances 4 lbs.; required the true weight, W, of the body and the lengths, a and b, of the arms.

Ans. $W =$ 6 lbs.; $a =$ 1 ft. 6 ins.; $b =$ 2 ft. 3 ins.

10. If a balance be false, having its arms in the ratio of 15 to 16, find how much per lb. a customer really pays for tea which is sold to him from the longer arm at 3s. 9d. per lb. *Ans.* 4s. per lb.

11. A straight uniform lever whose weight is 50 lbs. and length 6 feet, rests in equilibrium on a fulcrum when a weight of 10 lbs. is suspended from one extremity: find the position of the fulcrum and the pressure on it.

Ans. $2\frac{1}{2}$ ft. from the end at which 10 lbs. is suspended; 60 lbs.

12. On one arm of a false balance a body weighs 11 lbs.; on the other 17 lbs. 3 oz.; what is the true weight?

Ans. 13 lbs. 12 oz.

13. A bent lever is composed of two straight uniform rods of the same length, inclined to each other at 120°, and the fulcrum is at the point of intersection: if the weight of one rod be double that of the other, show that the lever will remain at rest with the lighter arm horizontal.

14. A uniform lever, l feet long, has a weight of W lbs., suspended from its extremity; find the position of the fulcrum when the long end of the lever balances the short

end with the weight attached to it, supposing each unit of length of the lever to be w lbs.

Ans. $\dfrac{l^2w}{2(W + lw)}$ is the short arm.

15. A lever, l ft. long, is balanced when it is placed upon a prop $\frac{1}{4}$ of its length from the thick end; when a weight of W lbs. is suspended from the small end the prop must be shifted $\frac{1}{2}$ ft. towards it in order to maintain equilibrium; required the weight of the lever. *Ans.* $\frac{1}{2}W$.

16. A lever, l ft. long; is balanced on a prop by a weight of W lbs.; first, when the weight is suspended from the thick end the prop is a ft. from it; secondly, when the weight is suspended from the small end the prop is b ft. from it; required the weight of the lever.

Ans. $\dfrac{W(a + b)}{l - (a + b)}$ lbs.

17. The forces, P and W, act at the arms, a and b, respectively, of a straight lever. When P and W make angles of 30° and 90° with the lever, show that when equilibrium takes place $P = \dfrac{2bW}{a}$.

18. Supposing the beam of a false balance to be uniform, a and b the lengths of the arms, P and Q the apparent weights, and W the true weight; when the weight of the beam is taken into account show that

$$\frac{a}{b} = \frac{P - W}{W - Q}.$$

19. If a be the length of the short arm in Ex. 14, what must be the length of the whole lever when equilibrium takes place?

Ans. $a + \sqrt{\dfrac{2aW}{w} + a^2}$.

20. A man whose weight is 140 lbs. is just able to support a weight that hangs over an axle of 6 ins. radius, by

hanging to the rope that passes over the corresponding wheel, the diameter of which is 4 ft.; find the weight supported. *Ans.* 560 lbs.

21. If the difference between the diameter of a wheel and the diameter of the axle be six times the radius of the axle, find the greatest weight that can be sustained by a force of 60 lbs. *Ans.* 240 lbs.

22. If the radius of the wheel is three times that of the axle, and the string round the wheel can support a weight of 40 lbs. only, find the greatest weight that can be lifted. *Ans.* 120 lbs.

23. What force will be required to work the handle of a windlass, the resistance to be overcome being 1156 lbs., the radius of the axle being six ins., and of the handle 2 ft. 8 ins.? *Ans.* 216.75 lbs.

24. Sixteen sailors, exerting each a force of 29 lbs., push a capstan with a length of lever equal to 8 ft., the radius of the capstan being 1 ft. 2 ins. Find the resistance which this force is capable of sustaining. *Ans.* 1 ton 8 cwt. 1 qr. $17\frac{5}{7}$ lbs.

25. Supposing them to have wound the rope round the capstan, so that it doubles back on itself, the radius of the axle is thus increased by the thickness of the rope. If this be 2 ins. how much will the power of the instrument be diminished. *Ans.* By $\frac{1}{8}$, or $12\frac{1}{2}$ per cent.

26. The radius of the axle of a capstan is 2 feet, and six men push each with a force of one cwt. on spokes 5 feet long; find the tension they will be able to produce in the rope which leaves the axle. *Ans.* 15 cwt.

27. The difference of the diameters of a wheel and axle is 2 feet 6 inches; and the weight is equal to six times the power; find the radii of the wheel and the axle. *Ans.* 18 ins.; 3 ins.

28. If the radius of a wheel is 4 ft., and of the axle 8 ins., find the power that will balance a weight of 500 lbs., the thickness of the rope coiled round the axle being one inch, the power acting without a rope.

Ans. 88.54 lbs.

29. Two given weights, P and Q, hang vertically from two points in the rim of a wheel turning on an axis; find the position of the weights when equilibrium takes place, supposing the angle between the radii drawn to the points of suspension to be 90°, and that θ is the angle which the radius, drawn to P's point of suspension, makes with the vertical.

Ans. $\tan\theta = \frac{Q}{P}$.

30. What weight can be supported on a plane by a horizontal force of 10 lbs., if the ratio of the height to the base is $\frac{3}{4}$?

Ans. $13\frac{1}{3}$ lbs.

31. The inclination of a plane is 30°, and a weight of 10 lbs. is supported on it by a string, bearing a weight at its extremity, which passes over a smooth pulley at its summit; find the tension in the string. *Ans.* 5 lbs.

32. The angle of a plane is 45°; what weight can be supported on it by a horizontal force of 3 lbs., and a force of 4 lbs. parallel to the plane, both acting together.

Ans. $3 + 4\sqrt{2}$ lbs.

33. A body is supported on a plane by a force parallel to it and equal to $\frac{1}{5}$ of the weight of the body; find the ratio of the height to the base of the plane.

Ans. $1 : 2\sqrt{6}$.

34. One of the longest inclined planes in the world is the road from Lima to Callao, in S. America; it is 6 miles long, and the fall is 511 ft. Calculate the inclination.

Ans. 55′ 27″, or 1 yard in 62.

35. If the force required to draw a wagon on a horizontal road be $\frac{1}{21}$th part of the weight of the wagon, what will be the force required to draw it up a hill, the slope of which is 1 in 43. *Ans.* $\frac{1}{14.11}$th part of the weight.

36. If the force required to draw a train of cars on a level railroad be $\frac{1}{200}$th part of the load, find the force required to draw it up a grade of 1 in 56.

Ans. $\frac{1}{43.75}$th part of the load.

37. What force is required (neglecting friction) to roll a cask weighing 964 lbs. into a cart 3 ft. high, by means of a plank 14 ft. long resting against the cart.

Ans. The force must exceed $206\frac{4}{7}$ lbs.

38. A body is at rest on a smooth inclined plane when the power, weight and normal pressure are 18, 26, and 12 lbs. respectively; find the inclination, α, of the plane to the horizon, and the angle, θ, which the direction of the power makes with the plane.

Ans. $\alpha = 37° 21' 26''$; $\theta = 28° 46' 54''$.

39. If the power which will support a weight when acting along the plane be half that which will do so acting horizontally, find the inclination of the plane. *Ans.* 60°.

40. A power P acting along a plane can support W, and acting horizontally can support x; show that

$$P^2 = W^2 - x^2.$$

41. A weight W would be supported by a power P acting horizontally, or by a power Q acting parallel to the plane; show that

$$\frac{1}{Q^2} = \frac{1}{P^2} + \frac{1}{W^2}$$

42. The base of an inclined plane is 8 ft., the height 6 ft., and $W = 10$ tons; required P and the normal pressure, N, on the plane.

Ans. $P = 6$ tons; $N = 8$ tons.

43. A weight is supported on an inclined plane by a force whose direction is inclined to the plane at an angle of 30°; when the inclination of the plane to the horizon is 30°, show that $W = P\sqrt{3}$.

44. A man weighing 150 lbs. raises a weight of 4 cwt. by a system of four movable pulleys arranged according to the second system; what is his pressure on the ground?

Ans. 122 lbs.

45. What power will be required in the second system with four movable pulleys to sustain a weight of 17 tons 12 cwt. *Ans.* 1 ton 2 cwt.

46. Two weights hang over a pulley fixed to the summit of a smooth inclined plane, on which one weight is supported, and for every 3 ins. that one descends the other rises 2 ins.; find the ratio of the weights, and the length of the plane, the height being 18 ins. *Ans.* 2 : 3; 27 ins.

47. If $W = 336$ lbs. and $P = 42$ lbs. in a combination of pulleys arranged according to the first system, how many movable pulleys are there? *Ans.* 4.

48. In a system of pulleys of the third kind in which there are 4 cords attached to the weight, determine the weight, W, supported, and the strain on the fixed pulley, the power being 100 lbs., and the weight, w, of each pulley 5 lbs.

Ans. $W = 15P + 11w = 1555$ lbs.; Strain $= 16P + 15w = 1675$ lbs.

49. In a system of pulleys of the third kind, there are 2 movable pulleys, each weighing $2\frac{1}{2}$ lbs. What power is required to support a weight of 6 cwt.? *Ans.* 94.57 lbs.

50. Find the power that will support a weight of 100 lbs. by means of a system of 4 pulleys, the strings being all attached to the weight, and each pulley weighing 1 lb.

Ans. $5\frac{4}{15}$ lbs.

51. The circumference of the circle corresponding to the point of application of P is 6 feet; find how many turns the screw must make on a cylinder 2 feet long, in order that W may be equal to $144P$. *Ans.* 48.

52. The distance between two consecutive threads of a screw is a quarter of an inch, and the length of the power arm is 5 feet; find what weight will be sustained by a power of 1 lb. *Ans.* 480π lbs.

53. How many turns must be given to a screw formed upon a cylinder whose length is 10 ins., and circumference 5 ins., that a power of 2 ozs. may overcome a pressure of 100 ozs.? *Ans.* 100.

54. A screw is made to revolve by a force of 2 lbs. applied at the end of a lever 3.5 ft. long; if the distance between the threads be $\frac{1}{2}$ in., what pressure can be produced? *Ans.* 9 cwts. 1 qr. 20 lbs.

55. The length of the power-arm is 15 inches; find the distance between two consecutive threads of the screw, that the mechanical advantage may be 30. *Ans.* π ins.

56. A weight of W pounds is suspended from the block of a single movable pulley, and the end of the cord in which the power acts, is fastened at the distance of b ft. from the fulcrum of a horizontal lever, a ft. long, of the second kind; find the force, P, which must be applied perpendicularly at the extremity of the lever to sustain W.

Ans. $P = \dfrac{Wb}{2a}$.

57. In a steelyard, the weight of the beam is 10 lbs., and the distance of its centre of gravity from the fulcrum is 2 ins., find where a weight of 4 lbs. must be placed to balance it. *Ans.* At 5 ins.

58. A body whose weight is $\sqrt{2}$ lbs., is placed on a rough plane inclined to the horizon at an angle of 45°. The coefficient of friction being $\frac{1}{\sqrt{3}}$, find in what direction a force of $(\sqrt{3} - 1)$ lbs. must act on the body in order just to support it. *Ans.* At an angle of 30° to the plane.

59. A rough plane is inclined to the horizon at an angle of 60°; find the magnitude and the direction of the least force which will prevent a body weighing 100 lbs. from sliding down the plane, the coefficient of friction being $\frac{1}{\sqrt{3}}$.

Ans. 50 lbs. inclined at 30° to the plane.

CHAPTER VIII.

THE FUNICULAR* POLYGON—THE CATENARY ATTRACTION.

130. Equilibrium of the Funicular Polygon.—If a cord whose weight is neglected, is suspended from two fixed points, A and B, and if a series of weights, P_1, P_2, P_3, etc., be suspended from the given points Q_1, Q_2, Q_3, etc., the cord will, when in equilibrium, form a polygon in a vertical plane, which is called the *Funicular Polygon.*

Let the tensions along the successive portions of the cord, AQ_1, Q_1Q_2, Q_2Q_3, etc., be respectively T_1, T_2, T_3, etc., and let θ_1, θ_2, θ_3, etc., be the inclinations of these portions to the horizon. Then Q_1 is in equilibrium under the action of three forces viz., P_1, acting vertically, T_1, the tension of the cord AQ_1, and T_2, the tension of Q_1Q_2. Resolving these forces we have,

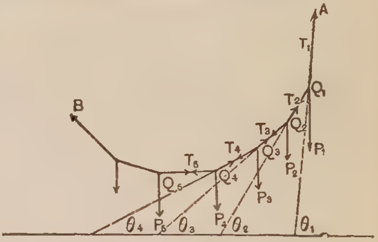

Fig. 69

for horizontal forces, $T_1 \cos \theta_1 - T_2 \cos \theta_2 = 0$, (1)

for vertical forces, $P_1 + T_2 \sin \theta_2 - T_1 \sin \theta_1 = 0$, (2)

In the same way for the point Q_2 we have,

for horizontal forces, $T_2 \cos \theta_2 - T_3 \cos \theta_3 = 0$, (3)

for vertical forces, $P_2 + T_3 \sin \theta_3 - T_2 \sin \theta_2 = 0$, (4)

---

* The term, *Funicular*, has reference alone to the cord, and has no mechanical significance.

Hence from (1) and (3) we have

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 = T_3 \cos\theta_3 = \text{etc.},$$

that is, *the horizontal components of the tensions in the different portions of the cord are constant.* Let this constant be denoted by T; then we have

$$T_1 = \frac{T}{\cos\theta_1}; \quad T_2 = \frac{T}{\cos\theta_2}; \quad T_3 = \frac{T}{\cos\theta_3}; \text{ etc.},$$

which in (2) and (4) give

$$P_1 + T\tan\theta_2 - T\tan\theta_1 = 0, \tag{5}$$

$$P_2 + T\tan\theta_3 - T\tan\theta_2 = 0, \tag{6}$$

and from (5) and (6) we have

$$\left.\begin{array}{ll} & \tan\theta_1 = \tan\theta_2 + \dfrac{P_1}{T}, \\ \text{and} & \tan\theta_2 = \tan\theta_3 + \dfrac{P_2}{T}. \\ \text{Similarly} & \tan\theta_3 = \tan\theta_4 + \dfrac{P_3}{T}, \\ \text{and} & \tan\theta_4 = \tan\theta_5 + \dfrac{P_4}{T}, \\ & \text{etc.,} \qquad \text{etc.} \end{array}\right\} \tag{7}$$

If we suppose the weights P_1, P_2, etc., each equal to W, (7) becomes

$$\tan\theta_1 - \tan\theta_2 = \tan\theta_2 - \tan\theta_3 = \tan\theta_3 - \tan\theta_4$$
$$= \ldots\ldots = \frac{W}{T}. \tag{8}$$

Hence, *the tangents of the successive inclinations form a series in Arithmetic Progression.* In the figure $\theta_5 = 0$,

$$\left.\begin{array}{l} \therefore\ \tan\theta_4 = \dfrac{W}{T};\quad \tan\theta_3 = \dfrac{2W}{T};\\ \tan\theta_2 = \dfrac{3W}{T};\ \tan\theta_1 = \dfrac{4W}{T};\ \text{etc.} \end{array}\right\} \qquad (9)$$

131. To Construct the Funicular Polygon when the Horizontal Projections of the successive Portions of the Cord are all equal.—Let Q_5Q_4, Q_4q_3, q_3q_2, q_2q_1, etc., be all of constant length $= a$, and let $Q_3q_3 = c$. Then since by (9) of Art. 130, the tangents of θ_4, θ_3, θ_2, θ_1, etc., are as 1, 2, 3, 4, etc., we have

$$Q_2m = 2Q_3q_3 = 2c;$$

$$Q_1n = 3Q_3q_3 = 3c;\ \text{etc.}$$

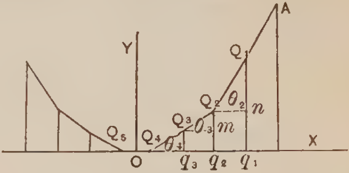

Fig. 70

Hence, taking the middle point, O, of the horizontal portion, Q_5Q_4, as origin, and the horizontal and vertical lines through it as axes of x and y, the co-ordinates of Q_3 are $(\frac{3}{2}a, c)$; those of Q_2 are $(\frac{5}{2}a, 3c)$; those of Q_1 are $(\frac{7}{2}a, 6c)$, and those of the nth vertex from Q_4 are evidently

$$x = \frac{2n+1}{2}\cdot a;\quad y = \frac{n(n+1)}{2}\cdot c.$$

Eliminating n from these equations we get

$$x^2 = \frac{2a^2y}{c} + \frac{a^2}{4} \qquad (1)$$

which, being independent of n, is satisfied by all the vertices indifferently, and is therefore the equation of a curve passing through all the vertices of the polygon, and denotes a parabola whose axis is the vertical line, OY, and whose vertex is vertically below O at a distance $= \dfrac{c}{8}$.

The shorter the distances Q_4Q_3, Q_3Q_2, etc., the more nearly does the *funicular polygon* coincide with the parabolic curve.

132. Cord Supporting a Load Uniformly Distributed over the Horizontal.—If the number of vertices of the polygon be very great, and the suspended weights all equal so that the load is distributed uniformly along the straight line, FE, the parabola which passes through all the vertices, virtually coincides with the cord or chain forming the polygon, and gives the figure of the *Suspension Bridge.* In this bridge the weights suspended from the successive portions of the chain are the weights of equal portions of the flooring. The weight of the chain itself and the weights of the sustaining bars are neglected in comparison with the weight of flooring and the load which it carries.

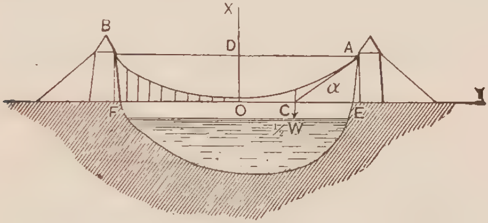

Fig. 71

Let the span, AB, $= 2a$, and the height, OD, $= h$. Then the equation of the parabola referred to the vertical and horizontal axes of x and y, respectively, through O, is

$$y^2 = 4mx, \tag{1}$$

$4m$ being the parameter.

Because the load between O and A is uniformly distributed over the horizontal, OE, its resultant bisects OE at C; therefore the tangents at A and O intersect at C (Art. 62).

From (1) we have

$$\frac{dy}{dx} = \frac{2m}{y} = \frac{y}{2x},$$

which is the tangent of the inclination of the curve at any point (x, y) to the axis of x. Hence the tangent at the point of support, A, makes with the horizon an angle, α, whose tangent is $\frac{2h}{a}$, which also is evident from the triangle ACE.

Let W be the weight on the cord; then $\frac{1}{2}W$ is the weight on OA, and therefore is the vertical tension, V, at A. Then the three forces at A are the vertical tension $V = \frac{1}{2}W$, the total tension at the end of the cord, acting along the tangent AC, and the horizontal tension, T, which is everywhere the same (Art. 130). Hence, by the triangle of forces (Art. 31) these forces will be represented by the three lines, AE, AC, CE, to which their directions are respectively parallel; therefore we have for the horizontal tension

$$T = \mathrm{AE} \cot \alpha = W\frac{a}{4h},$$

and the total tension at A is

$$\sqrt{V^2 + T^2} = \frac{W}{4h}\sqrt{4h^2 + a^2}.$$

EXAMPLE.

The entire load on the cord in (Fig. 71) is 320000 lbs.; the span is 150 ft. and the height is 15 ft.; find the tension at the points of support and at the lowest point and also the inclination of the curve to the horizon at the points of support.

$$\tan \alpha = \frac{2h}{a} = .4; \quad \therefore \quad \alpha = 21° \, 48'.$$

The vertical tension at each point of support is

$$V = \tfrac{1}{2} \text{ weight} = 160000 \text{ lbs.};$$

the horizontal tension is

$$T = W\frac{a}{4h} = 400000 \text{ lbs.};$$

and the total tension at one end is

$$\sqrt{V^2 + T^2} = 430813 \text{ lbs.}$$

133. The Common Catenary.—Its Equation.—A *catenary* is the curve assumed by a perfectly flexible cord when its ends are fastened at two points, A and B, nearer together than the length of the cord. When the cord is of constant thickness and density, *i. e.*, when equal portions of it are equally heavy, the curve is called the *Common Catenary*, which is the only one we shall consider.

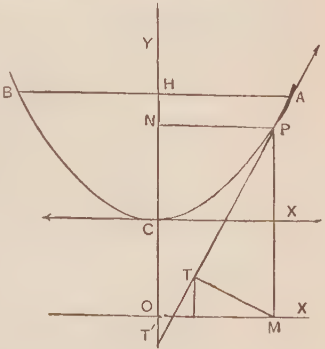

Fig. 72

Let A and B be the fixed points to which the ends of the cord are attached; the cord will rest in a *vertical plane* passing through A and B, which may be taken to be the plane of the paper Let C be the lowest point of the catenary; take this as the origin of co-ordinates, and let the horizontal line through C be taken for the axis of x, and the vertical line through C for the axis of y. Let (x, y) be any point, P, in the curve; denote the length of the arc, CP, by s; let c* be the length of the cord whose weight is equal to the tension at C; and T the length of the cord whose weight is equal to the tension at P.

---

* The weight of a unit of length of the cord being here taken as the unit of weight.

Then the arc, CP, after it has assumed its permanent form of equilibrium, may be considered as a rigid body kept at rest by three forces, viz.: (1) T, the tension, acting at P along the tangent, (2) c, the horizontal tension at the lowest point C, and (3) the weight of the cord, CP, acting vertically downward, and denoted by s. Draw PT' the tangent at P, meeting the axis of y at T'. Then by the triangle of forces (Art. 31), these forces may be represented by the three lines PT', NP, $T'N$, to which their directions are respectively parallel. Therefore

$$\frac{T'N}{NP} = \frac{\text{weight of } CP}{\text{tension at } C},$$

or

$$\frac{dy}{dx} = \frac{s}{c}. \tag{a}$$

Differentiating, substituting the value of ds, and reducing, we have

$$\frac{d\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{dx}{c}.$$

Integrating, and remembering that when $x = 0$, $\frac{dy}{dx} = 0$, we obtain

$$\log\left[\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}\right] = \frac{x}{c};$$

therefore

$$\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2} = e^{\frac{x}{c}},$$

where e is the Naperian base. Solving this equation for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = \tfrac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right); \tag{1}$$

and by integration, observing that $y = 0$ when $x = 0$, we have

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) - c, \tag{2}$$

which is the equation required. We may simplify this equation by moving the origin to the point, O, at a distance equal to c below C, by putting $y - c$ for y, so that (2) becomes,

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right), \tag{3}$$

which *is the equation of the catenary, in the usual form.* The horizontal line through O is called *the directrix* of the catenary,* and O is called *the origin.*

Cor. 1.—To find the length of the arc, CP, we have

$$ds = \sqrt{1 + \frac{dy^2}{dx^2}}\,dx$$

$$= \sqrt{1 + \tfrac{1}{4}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)^2}\,dx, \text{ from (1)},$$

$$= \tfrac{1}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)dx; \tag{4}$$

$$\therefore\ s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) \tag{5}$$

the constant being $= 0$, since when $x = 0$, $s = 0$.

This equation may also be found immediately by equating the values of $\frac{dy}{dx}$ in (a) and (1).

---

* See Price's Anal. Mechs., Vol. I, p. 216.

Cor. 2.—Since $c = OC$ is the length of the cord whose weight is equal to the tension of the curve at the lowest point, C, it follows that, if the half, BC, of the curve were removed, and a cord of length c, and of the same thickness and density as the cord of the curve, were joined to the arc CP, and suspended over a smooth peg at C, the curve would be in equilibrium.

Cor. 3.—We have from the triangle, PNT',

$$\frac{\text{tension at } P}{\text{tension at } C} = \frac{PT'}{PN},$$

or

$$\frac{T}{c} = \frac{ds}{dx} = \frac{y}{c} \text{ from (3) and (4),}$$

$$\therefore \quad T = y;$$

that is, *the tension at any point of the catenary is equal to the weight of a portion of the cord whose length is equal to the ordinate at that point.*

Therefore if a cord of constant thickness and density hangs freely over any two smooth pegs, the vertical portions which hang over the pegs, must each terminate on the directrix of the catenary.

Cor. 4.—From (3) and (5) we have

$$y^2 = s^2 + c^2, \tag{6}$$

and from (6) we have

$$s = y\frac{dy}{ds}. \tag{7}$$

At the point, P, draw the ordinate, PM, and from M, the foot of the ordinate, draw the perpendicular MT. Then

$$PT = y \cos MPT = y\frac{dy}{ds},$$

which in (7) gives

$$PT = s = \text{the arc, } CP, \tag{8}$$

and since $y^2 = PT^2 + TM^2$, we have from (6) and (8)

$$TM = c. \tag{9}$$

Therefore the point, T, is on the involute of the catenary which originates from the curve at C, TM is a tangent to this involute, and TP, the tangent to the catenary, is normal to the involute, (See Calculus, Art. 124). As TM is the tangent to this last curve, and is equal to the constant quantity, c, the involute is the equitangential curve, or tractrix (See Calculus, p. 357).

By means of (8) and (9) we may construct the *origin* and *directrix* of the catenary as follows: *On the tangent at any point, P, measure off a length, PT, equal to the arc, CP; at T erect a perpendicular, TM, to the tangent meeting the ordinate of P at M; then the horizontal line through M is the directrix, and its intersection with the axis of the curve is the origin.*

Cor. 5.—Combining (2) and (5) we obtain

$$(y + c)^2 = s^2 + c^2,$$

therefore

$$s^2 = y^2 + 2cy. \tag{10}$$

The catenary possesses many interesting geometric and mechanical properties, but a discussion of them would carry us beyond the limits of this treatise. The student who wishes to pursue the subject further, is referred to Price's Anal. Mechs., Vol. I, and Minchin's Statics.

133a. Attraction of a Spherical Shell.—By the law of universal gravitation every particle of matter attracts every other particle with a force that varies *directly* as the mass of the attracting particle, and *inversely* as the square of the distance between the particles.

To find the resultant attraction of a spherical shell of uniform density and small uniform thickness, on a particle.

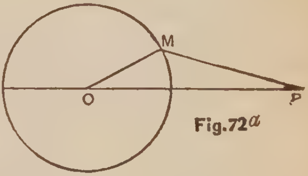

Fig. 72*a*

(1) Suppose the particle, P, on which the value of the attraction is required, to be outside the shell.

Let ρ and k be the density and thickness of the shell, O its centre, and M any particle of it. Let $OM = a$, $PM = r$, $OP = c$, the angle $MOP = \theta$, ϕ the angle which the plane MOP makes with a fixed plane through OP.

Then the mass of the element at M (Art. 88) is $\rho k\, a^2 \sin\theta\, d\theta\, d\phi$. The attraction of the whole shell acts along OP; the attraction of the elementary mass at M on P in the direction PM

$$= \frac{\rho k\, a^2 \sin\theta\, d\theta\, d\phi}{r^2};$$

therefore the attraction of M on P, resolved along OP,

$$= \frac{\rho k\, a^2 \sin\theta\, d\theta\, d\phi}{r^2}\, \frac{c - a\cos\theta}{r}. \tag{1}$$

We shall eliminate θ from this equation by means of

$$r^2 = a^2 + c^2 - 2ac\cos\theta;$$

$$\therefore\ rdr = ac\sin\theta\, d\theta;$$

$$\therefore \quad \sin\theta\, d\theta = \frac{rdr}{ac},$$

and $$c - a\cos\theta = \frac{r^2 + c^2 - a^2}{2c};$$

substituting these values in (1), the attraction of M on P along PO

$$= \frac{\rho k a}{2c^2}\left(1 + \frac{c^2 - a^2}{r^2}\right) dr\, d\phi. \qquad (2)$$

To obtain the resultant attraction of the whole shell, we take the ϕ-integral between the limits 0 and 2π, and the r-integral between $c - a$ and $c + a$.

Hence the resultant attraction of the shell on P along PO

$$= \frac{\rho k a}{2c^2}\int_{c-a}^{c+a}\int_0^{2\pi}\left(1 + \frac{c^2 - a^2}{r^2}\right) dr\, d\phi,$$

$$= \frac{\pi \rho k a}{c^2}\int_{c-a}^{c+a}\left(1 + \frac{c^2 - a^2}{r^2}\right) dr,$$

$$= \frac{4\pi\rho k a^2}{c^2} = \frac{\text{mass of the shell}}{c^2}. \qquad (3)$$

Since c is the distance of the point P from the centre this shows that the attraction of the shell on the particle at P is the same as if the mass of the shell were condensed into its centre.

It follows from this that a sphere which is either homogeneous or consists of concentric spherical shells of uniform density, attracts the particle at P in the same manner as if the whole mass were collected at its centre.

(2) Let the particle, P, be inside the sphere. Then we proceed exactly as before, and obtain equation (2), which is true whether the particle be outside or inside the sphere:

but the r-limits in this case are $a - c$ and $a + c$. Hence from (2) we have, by performing the ϕ-integration,

$$\text{attraction of shell} = \frac{\pi\rho ka}{c^2}\int_{a-c}^{a+c}\left(1 - \frac{a^2 - c^2}{r^2}\right)dr,$$

$$= \frac{\pi\rho ka}{c^2}(2c - 2c) = 0.$$

therefore a particle within the shell is equally attracted in every direction, *i. e.*, is not attracted at all.

Cor.—If a particle be inside a homogenous sphere at the distance r from its centre, all that portion of the sphere which is at a greater distance from the centre than the particle produces no effect on the particle, while the remainder of the sphere attracts the particle in the same manner as if the mass of the remainder were all collected at the centre of the sphere. Thus the attraction of the sphere on the particle

$$= \frac{\frac{4}{3}\pi\rho r^3}{r^2} \quad \text{or} \quad \frac{4\pi\rho r}{3}.$$

Hence, *within a homogeneous sphere the attraction varies as the distance from the centre.*

The propositions respecting the attraction of a uniform spherical shell on an external or internal particle were given by Newton (Principia, Lib. I, Prop. 70, 71). (See Todhunter's Statics, p. 275, also Pratt's Mechs., p. 137, Price's Anal. Mechs., Vol. I, p. 266, Minchin's Statics, p. 403).

EXAMPLES.

1. The span $AB = 800$ feet, and $CO = 1600$ feet, find the length of the curve, CA, the height, CH, and the

inclination, θ, of the curve to the horizon at either point of suspension.

(1) Here $\frac{x}{c} = \frac{1}{4}$, and $e = 2 \cdot 71828$,

therefore $$e^{\frac{x}{c}} = (2 \cdot 71828)^{\frac{1}{4}} = 1 \cdot 2840,$$

and $$e^{-\frac{x}{c}} = (2 \cdot 71828)^{-\frac{1}{4}} = 0 \cdot 7788.$$

Substituting these values in (5) we get

$$S = 800 \times 0 \cdot 5052 = 404 \cdot 16.$$

Hence $$CA = 404 \cdot 16 \text{ feet.}$$

(2) $$CH = y - c = \frac{c}{2}\left(e^{\frac{1}{4}} + e^{-\frac{1}{4}}\right) - c$$

$$= 800 \times 2 \cdot 0628 - 1600$$

$$= 50 \cdot 24 \text{ feet.}$$

(3) $$\tan\theta = \frac{dy}{dx} = \tfrac{1}{2}\left(e^{\frac{1}{4}} - e^{-\frac{1}{4}}\right), \text{ from (1);}$$

$$= 0 \cdot 2526,$$

therefore $$\theta = 14^\circ\, 11'.$$

Otherwise $\tan\theta = \frac{s}{c}$, from (a), $= \frac{404 \cdot 16}{1600} = 0 \cdot 2526$, as before.

2. The entire load on the cord in Fig. 71 is 160000 lbs., the span is 192 ft., and the height is 15 ft.; find the tension at the points of support, and also the tension at the lowest point. *Ans.* Tension at one end = 268208 lbs.
Horizontal tension = 256000 "

3. A chain, ACB, 10 feet long, and weighing 30 lbs., is suspended so that the height, $CH, = 4$ feet; find the horizontal tension, and the inclination, θ, of the chain to the horizon at the points of support.

Ans. Horizontal tension $= 3\frac{3}{8}$ lbs., $\theta = 77° \; 19'$.

4. A chain 110 ft. long is suspended from two points in the same horizontal plane, 108 ft. apart; show that the tension at the lowest point is 1.477 times the weight of the chain nearly.

PART II.

KINEMATICS (MOTION).

CHAPTER I.

RECTILINEAR MOTION.

134. Definitions. — Velocity. — Kinematics is that branch of Dynamics which treats of motion without reference to the *bodies* moved or the *forces* producing the motion (Art. 1). Although we do not know motion as free from *force* or from the *matter* that is moved, yet there are cases in which it is advantageous to separate the ideas of force, matter, and motion, and to study motion in the abstract, *i. e.*, without any reference to *what is moving*, or the *cause of motion.* To the study of pure motion, then, we devote this and the following chapter.

The velocity of a particle has been defined to be *its rate of motion* (Art. 6). The formulæ for uniform and variable velocities are those which were deduced in Art. 7. From (1) and (2) of that Art. we have

$$v = \frac{s}{t}; \tag{1}$$

$$v = \frac{ds}{dt}; \tag{2}$$

in which v is the velocity, s the space, and t the time.

EXAMPLES.

n. g 1. A body moves at the rate of 754 yards per hour. Find the velocity in feet per second.

Since the velocity is uniform we use (1), hence

$$v = \frac{s}{t} = \frac{754 \times 3}{60 \times 60} = 0.628 \text{ ft. per sec., } \textit{Ans.}$$

n. g 2. Find the position of a particle at a given time, t, when the velocity varies as the distance from a given point on the rectilinear path.

Here the velocity being variable we have from (2)

$$v = \frac{ds}{dt} = ks,$$

where k is a constant;

$$\text{therefore} \qquad \frac{ds}{s} = k\,dt\,; \quad \therefore \quad \log s = kt + c, \qquad (1)$$

where c is an arbitrary constant.

Now if we suppose that s_0 is the distance of the particle from the given point when $t = 0$ we have $c = \log s_0$, which in (1) gives

$$\log \frac{s}{s_0} = kt\,; \quad \text{or} \quad s = s_0 e^{kt}.$$

3. A railway train travels at the rate of 40 miles per hour; find its velocity in feet per second.

Ans. 58.66 ft. per second.

4. A train takes 7 h. 31 m. to travel 200 miles; find its velocity. *Ans.* 39.02 ft. per sec.

5. If $s = 4t^3$, find the velocity at the end of five seconds.

Ans. 300 ft. per sec.

6. Find the position of the particle in Ex. 2, when the velocity varies as the time. *Ans.* $s = s_0 + \frac{1}{2}kt^2$.

7. Find the distance the particle will move in one minute, when the velocity is 10 ft. at the end of one second and varies as the time. *Ans.* 18000 ft.

135. Acceleration.—Acceleration has been defined to be the *rate of change of velocity* (Art. 8). It is a *velocity increment.* The formulæ for acceleration are from (1), (2), and (3) of (Art. 9),

$$f = \frac{v}{t}; \tag{1}$$

$$f = \frac{dv}{dt}; \tag{2}$$

$$f = \frac{d^2s}{dt^2}; \tag{3}$$

(1) being for uniform, and (2) and (3) for variable, acceleration.

If the velocity *decreases,* f is negative, and (2) and (3) become

$$\frac{dv}{dt} = -f; \quad \frac{d^2s}{dt^2} = -f;$$

and the velocity and time are inverse functions of each other.

136. The Relation between the Space and Time when the Acceleration = 0.

Here we have

$$\frac{d^2s}{dt^2} = 0,$$

so that if v_0 is the constant velocity we have

$$\frac{ds}{dt} = v_0;$$

$$\therefore \quad s = v_0 t + s_0;$$

in which s_0 is the space which the body has passed over when $t = 0$. If t is computed from the time the body starts from rest, then $s = v_0 t$. The student will observe that this is a case of uniform velocity.

137. The Relation (1) between the Space and Time, and (2) between the Space and Velocity, when the Acceleration is Constant.

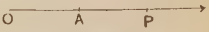

Fig.73

(1) Let A be the initial position of the particle supposed to be moving toward the right, P its position at any time, t, from A, v its velocity at that time, and f the constant acceleration of its velocity. Take any fixed point, O, in the line of motion as origin, and let $OA = s_0$; $OP = s$. Then the equation of motion is

$$\frac{d^2s}{dt^2} = f; \tag{1}$$

$$\therefore \quad \frac{ds}{dt} = ft + c.$$

Suppose the velocity of the particle, at the point A to be v_0, then when $t = 0$, $v = v_0$;[*] hence $c = v_0$, and

$$v = \frac{ds}{dt} = ft + v_0; \tag{2}$$

$$\therefore \quad s = \tfrac{1}{2}ft^2 + v_0 t + c'.$$

But when $t = 0$, $s = s_0$; hence $c' = s_0$, and

$$s = \tfrac{1}{2}ft^2 + v_0 t + s_0, \tag{3}$$

Hence if a particle moves from rest from the origin O, with a constant acceleration, we have

---

[*] Called *initial* velocity and space respectively, or the velocity the particle has, and space it has moved over at the instant t begins to be reckoned.

$$s = \tfrac{1}{2}ft^2, \tag{4}$$

and thus the space described varies as the square of the time.

(2) From (1) we have

$$ds \frac{d^2s}{dt^2} = f ds;$$

$$\therefore \quad \frac{ds^2}{dt^2} = 2fs + C.$$

But when $s = s_0$, $v = v_0$; hence $C = v_0{}^2 - 2fs_0$, and therefore

$$v^2 = 2fs + v_0{}^2 - 2fs_0. \tag{5}$$

Equations (2) and (3) give the velocity and position of the particle in terms of t; and (5) gives the velocity in terms of s.

138. When the Acceleration Varies directly as the Time from a State of Rest, find the Velocity and Space at the end of the Time *t*.

Here $$\frac{d^2s}{dt^2} = at;$$

$$\therefore \quad \frac{ds}{dt} = \tfrac{1}{2}at^2 + v_0,$$

where v_0 is the initial velocity;

$$\therefore \quad s = \tfrac{1}{6}at^3 + v_0 t,$$

the initial space being 0 since t is estimated from rest.

139. When the Acceleration Varies directly as the Distance from a given Point in the line of Motion, and is negative, find the Relation between the Space and Time.

Here
$$\frac{d^2s}{dt^2} = -ks;$$

$$\therefore \quad 2ds\frac{d^2s}{dt^2} = -2ks\,ds;$$

$$\frac{ds^2}{dt^2} = k(s_0^2 - s^2),$$

by calling s_0 the value of s when the particle is at rest.

$$\therefore \quad -\frac{ds}{\sqrt{s_0^2 - s^2}} = k^{\frac{1}{2}}dt,$$

the negative sign being taken since the particle is moving towards the origin;

$$\therefore \quad \cos^{-1}\frac{s}{s_0} = k^{\frac{1}{2}}t,$$

if $s = s_0$ when $t = 0$;

$$\therefore \quad s = s^0 \cos k^{\frac{1}{2}}t.$$

EXAMPLES.

1. A body commences to move with a velocity of 30 ft. per sec., and its velocity is increased in each second by 10 ft. Find the space described in 5 seconds.

Here $f = 10$, $v_0 = 30$, $s_0 = 0$, and $t = 5$, therefore from (3) we have

$$s = \tfrac{1}{2}\cdot 10 \cdot 25 + 30 \cdot 5 = 275, \textit{ Ans.}$$

2. A body starting with a velocity of 10 ft. per sec., and moving with a constant acceleration, describes 90 ft. in 4 secs.; find the acceleration. *Ans.* $6\frac{1}{4}$ ft. per sec.

3. Find the velocity of a body which starting from rest with an acceleration of 10 ft. per sec., has described a space of 20 ft. *Ans.* 20 ft.

4. Through what space must a body pass under an acceleration of 5 ft. per sec., so that its velocity may increase from 10 ft. to 20 ft. per sec.? *Ans.* 30 ft.

5. In what time will a body moving* with an acceleration of 25 ft. per sec., acquire a velocity of 1000 ft. per second? *Ans.* 40 secs.

6. A body starting from rest has been moving for 5 minutes, and has acquired a velocity of 30 miles an hour; what is the acceleration in feet per second?

Ans. $\frac{11}{75}$ ft. per sec.

7. If a body moves from rest with an acceleration of $\frac{2}{3}$ ft. per sec., how long must it move to acquire a velocity of 40 miles an hour? *Ans.* 88 secs.

140. Equations of Motion for Falling Bodies.—The most important case of the motion of a particle with a constant acceleration in its line of motion is that of a body moving under the action of gravity, which for small distances above the earth's surface may be considered constant. When a body is allowed to fall freely, it is found to acquire a velocity of about 32.2 feet per second during every second of its motion, so that it moves with an acceleration of 32.2 feet per second (Art. 21). This acceleration is less at the summit of a high mountain than near the surface of the earth; and less at the equator than in the neighborhood of the poles; *i. e.*, the velocity which a body acquires in falling freely for one second varies with the *latitude* of the place, and with its *altitude* above the sea level; but is independent of the *size* of the body and of its *mass*. Practically, however, bodies do not *fall freely*, as the resistance of the air opposes their motion, and therefore in practical cases at high speed (*e. g.*, in artillery) the resistance of the air must be taken into account. But at present we shall neglect this resistance, and consider the bodies as moving in vacuo under the action of gravity, *i. e.*, with a constant acceleration of about 32.2 feet per second.

---

* In each case the body is supposed to start from rest unless otherwise stated.

As neither the *substance* of the body nor the *cause* of the motion needs to be taken into consideration, all problems relating to falling bodies may be regarded as cases of accelerated motion, and treated from purely geometric considerations. Therefore if we denote the acceleration by g, as in Art. 23, and consider the particle in Art. 137 to be moving vertically downwards, then (2), (3), (5) of Art. 137 become, by substituting g for f,

$$\left.\begin{aligned} v &= gt + v_0, \\ s &= \tfrac{1}{2}gt^2 + v_0t + s_0, \\ v^2 &= 2gs + v_0^2 - 2gs_0, \end{aligned}\right\} \qquad \text{(A)}$$

s being measured as before from a fixed point, O, in the line of motion.

Suppose the particle to be projected downward from O, then A commences with O and $s_0 = 0$. Hence (A) becomes

$$v = gt + v_0, \qquad (1)$$

$$s = \tfrac{1}{2}gt^2 + v_0t, \qquad (2)$$

$$v^2 = 2gs + v_0^2. \qquad (3)$$

As a particular case suppose the particle to be dropped from rest at O (Fig. 73). Then A coincides with O, and $s_0 = 0$, $v_0 = 0$. Hence equations (A) become

$$v = gt, \qquad (4)$$

$$s = \tfrac{1}{2}gt^2, \qquad (5)$$

$$v^2 = 2gs. \qquad (6)$$

141. When the Particle is Projected Vertically Upwards.—Here if we measure s upwards from the point of projection, O. the acceleration tends to diminish the space and therefore the acceleration is negative, and the equation of motion is (Art. 135)

$$\frac{d^2s}{dt^2} = -g.$$

In other respects the solution is the same. Taking therefore $s_0 = 0$ in (A) and changing the sign of g,* we obtain

$$v = v_0 - gt, \tag{1}$$

$$s = v_0 t - \tfrac{1}{2}gt^2, \tag{2}$$

$$v^2 = v_0^2 - 2gs. \tag{3}$$

COR. 1.—*The time during which a particle rises when projected vertically upwards.*

When the particle reaches its highest point, its velocity is zero. If therefore we put $v = 0$ in (1), the corresponding value of t will be the time of the particle ascending to a state of rest.

$$\therefore\ t = \frac{v_0}{g}.$$

COR. 2.—*The time of flight before returning to the starting point.*

From (2) we have the distance of the particle from the starting point after t seconds, when projected vertically upwards with the velocity v_0. Now when the particle has risen to its maximum height and returned to the point of projection, $s = 0$. If, therefore, we put $s = 0$ in (2), and solve for t, we shall get the time of flight. Therefore,

---

* g is positive or negative according as the particle is descending or ascending.

$$v_0t - \tfrac{1}{2}gt^2 = 0;$$

which gives $$t = 0, \quad \text{or} \quad \frac{2v_0}{g}.$$

The first value of t shows the time before the particle starts, the latter shows the time when it has returned. Hence, the *whole time* of flight is $\frac{2v_0}{g}$, which is just double the time of rising (Cor. 1); that is, *the time of rising equals the time of falling.*

The final velocity, by (1) of Art. 140, $= gt = g \times \frac{v_0}{g}$ (Cor. 1) $= v_0$; hence a body returns to any point in its path with the same velocity at which it left it. In other words, a body passes each point in its path with the same velocity, whether rising or falling, since the velocity at any point may be considered as a velocity of projection.

Cor. 3.—*The greatest height to which the particle will rise.*

At the summit $v = 0$, and the corresponding value of s will be the greatest height to which the particle will rise; when $v = 0$, (3) becomes

$$v_0^2 = 2gs;$$

$$\therefore \quad s = \frac{v_0^2}{2g}.$$

Cor. 4.—Since $v_0^2 = 2gs$, where s is the height from which a body falls to gain the velocity v_0, it follows that a body will rise through the same space in losing a velocity v_0 as it would fall through to gain it.

EXAMPLES.

1. A body projected vertically downwards with a velocity of 20 ft. a sec. from the top of a tower, reaches the ground in 2.5 secs.; find the height of the tower.

Here $t = 2\frac{1}{2}$, and $v_0 = 20$; assume $g = 32$. Then from (2) of Art. 140 we have

$$s = \tfrac{16 \times 25}{4} + 20 \times \tfrac{5}{2} = 150 \text{ ft.}$$

2. A body is projected vertically upwards with a velocity of 200 ft. per second; find the velocity with which it will pass a point 100 ft. above the point of projection.

Here $v_0 = 200$, $s = 100$; therefore from (3) we have

$$v^2 = 40000 - 6400 = 33600;$$

$$\therefore \quad v = 40\sqrt{21}.$$

3. A man is ascending in a balloon with a uniform velocity of 20 ft. per sec., when he drops a stone which reaches the ground in 4 secs.; find the height of the balloon.

Here $v_0 = 20$, and $t = 4$; therefore from (2) we have, after changing the sign of the second member to make the result positive,

$$s = -(80 - 256) = 176,$$

which was the height of the balloon.

4. A body is projected upwards with a velocity of 80 ft.; after what time will it return to the hand?

Ans. 5 seconds.

5. With what velocity must a body be projected vertically upwards that it may rise 40 ft.?

Ans. $16\sqrt{10}$ ft. per sec.

6. A body projected vertically upwards passes a certain point with a velocity of 80 ft. per sec.; how much higher will it ascend? *Ans.* 100 ft.

7. Two balls are dropped from the top of a tower, one of them 3 secs. before the other; how far will they be apart 5 secs. after the first was let fall? *Ans.* 336 ft.

8. If a body after having fallen for 3 secs. breaks a pane of glass and thereby loses one-third of its velocity, find the entire space through which it will have fallen in 4 secs. *Ans.* 224 ft.

142. Composition of Velocities.—(1) From the *Parallelogram of Velocities*, (Art. 29, Fig. 2), we see that if AB represents in magnitude and direction the space which would be described in one second by a particle moving with a given velocity, and AC represents in magnitude and direction the space which would be described in one second by another particle moving with its velocity, then AD, *the diagonal of the parallelogram, represents the resultant velocity in magnitude and direction.*

(2) Hence *the resultant of any two velocities, as* AB, BD, (Fig. 2), *is a velocity represented by the third side,* DA, *of the triangle* ABD; *and if a point have simultaneously, velocities represented by* AB, BC, CA, *the sides of a triangle, taken in the same order, it is at rest.*

The lines which are taken to represent any given *forces* may clearly be taken to represent the *velocities* which measure these forces (Art. 19), therefore from the *Polygon* and *Parallelopiped of Forces* the *Polygon and Parallelopiped of Velocities follow.*

(3) Hence, *if any number of velocities be represented in magnitude and direction by the sides of a closed polygon, taken all in the same order, the resultant is zero.*

(4) Also, *if three velocities be represented in magnitude*

and direction by the three edges of a parallelopiped, the resultant velocity will be represented by the diagonal.

(5) When there are two velocities or three velocities in two or in three rectangular directions, the resultant is the square root of the sum of their squares. Thus, if $\frac{ds}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, are the velocities of the moving point and its components parallel to the axes, we have from (2) of Art. 30,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}, \tag{1}$$

and from (1) of Art. 34,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}. \tag{2}$$

143. Resolution of Velocities.—As the diagonal of the parallelogram (Fig. 2), whose sides represent the component velocities was found to represent the resultant velocity, so any velocity, represented by a given straight line, may be resolved into component velocities represented by the sides of the parallelogram of which the given line is the diagonal.

It will be easily seen that (2) of Art. 134 is equally applicable whether the point be considered as moving in a straight line or in a curved line; but since in the latter case the direction of motion continually changes, the mere *amount* of the velocity is not sufficient to describe the motion completely, so it will be necessary to know at every instant the *direction*, as well as the *magnitude*, of the point's velocity. In such cases as this the method commonly employed, whether we deal with velocities or accelerations, consists mainly in studying, not the velocity or acceleration, *directly*, but its components parallel to any three assumed rectangular axes. If the particle be at the point (x, y, z), at the time t, and if we denote its velocities parallel respectively to the three axes by u, v, w, we have

$$\frac{dx}{dt} = v_x; \quad \frac{dy}{dt} = v_y; \quad \frac{dz}{dt} = v_z.$$

Denoting by v the velocity of the moving particle along the curve at the time t, we have as above

$$v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}, \tag{1}$$

and if α, β, γ be the angles which the direction of motion along the curve makes with the axes, we have, as in (2) of (Art. 34),

$$\frac{dx}{dt} = \frac{ds}{dt}\cos\alpha = v\cos\alpha = v_x;$$

$$\frac{dy}{dt} = \frac{ds}{dt}\cos\beta = v\cos\beta = v_y$$

$$\frac{dz}{dt} = \frac{ds}{dt}\cos\gamma = v\cos\gamma = v_z.$$

Hence each of the components $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, is to be found from the whole velocity by *resolving the velocity*, *i. e.*, by multiplying the velocity by *cosine of the angle* between the direction of motion and that of the component.

EXAMPLES.

1. A body moves under the influence of two velocities, at right angles to each other, equal respectively to 17.14 ft. and 13.11 ft. per second. Find the magnitude of the resultant motion, and the angles into which it divides the right angle.

Ans. 21.579 ft. per sec.; 37° 25′ and 52° 35′.

2. A ship sails due north at the rate of 4 knots per hour, and a ball is rolled towards the east, across her deck, at right angles to her motion at the rate of 10 ft. per second. Find the magnitude and direction of the real motion of the ball.

Ans. 12.07 ft. per sec.; and N. 56° E.

3. A boat moves N. 30° E., at the rate of 6 miles per hour. Find its rate of motion northerly and easterly.

Ans. 5.2 miles per hour north, and 3 miles per hour east.

144. Motion on an Inclined Plane.—By an extension of the equations of Art. 140, we may treat the case of a particle sliding from rest down a smooth inclined plane. As this is a very simple case in which an acceleration is resolved, it is convenient to treat of it in this part of our work; yet as it properly belongs to the theory of constrained motion, we are unable to give a complete solution of it, until the principles of such motion have been explained in a future chapter.

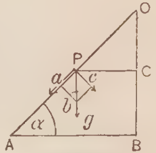

Fig. 74.

Let P be the position of the particle at any time, t, on the inclined plane OA, OP $= s$, its distance from a fixed point, O, in the line of motion, and let α be the inclination of OA to the horizontal line AB. Let Pb represent g, the vertical acceleration with which the body would move if free to fall. Resolve this into two components, P$a = g \sin \alpha$ along, and P$c = g \cos \alpha$ perpendicular to OA. The component $g \cos \alpha$ produces pressure on the plane, but does not affect the motion. The only acceleration down the plane is that component of the whole acceleration which is parallel to the plane, viz., $g \sin \alpha$. The equation of motion, therefore, is

$$\frac{d^2s}{dt^2} = g \sin \alpha, \qquad (1)$$

the solution of which, as $g \sin \alpha$ is constant, is included in that of Art. 140; and all the results for particles moving vertically as given in Arts. 140 and 141 will be made to apply to (1) by writing $g \sin \alpha$ for g. Thus, if the particle be projected down or up the plane, we get from (1), (2), (3) of Arts. 140 and 141, by this means

$$v = v_0 \pm g \sin \alpha \cdot t, \tag{2}$$

$$s = v_0 t \pm \tfrac{1}{2} g \sin \alpha \cdot t^2, \tag{3}$$

$$v^2 = v_0^2 \pm 2g \sin \alpha \cdot s, \tag{4}$$

in which the + or − sign is to be taken according as the body is projected *down* or *up* the plane.

If the particle starts from rest from 0, we get from (4), (5), (6) of Art. 140

$$v = g \sin \alpha \cdot t, \tag{5}$$

$$s = \tfrac{1}{2} g \sin \alpha \cdot t^2, \tag{6}$$

$$v^2 = 2g \sin \alpha \cdot s. \tag{7}$$

Cor. 1.—*The velocity acquired by a particle in falling down a given inclined plane.*

Draw PC parallel to AB (Fig. 74), then if v be the velocity at P, we have from (7)

$$\begin{aligned} v^2 &= 2g \sin \alpha \cdot s \\ &= 2g \cdot \mathrm{OC}. \end{aligned}$$

Hence, from (6) of Art. 140 the velocity is the same at *P* as if the particle had fallen through the vertical space OC; that is, *the velocity acquired in falling down a smooth inclined plane is the same as would be acquired in falling freely through the perpendicular height of the plane.*

Cor. 2.—*When the particle is projected up the plane with a given velocity, to find how high it will ascend, and the time of ascent.*

From (4) we have

$$v^2 = v_0^2 - 2g \sin \alpha \cdot s.$$

When $v = 0$ the particle will stop; hence, the distance it will ascend will be given by the equation

$$0 = v_0^2 - 2g \sin \alpha \cdot s,$$

$$\therefore \quad s = \frac{v_0^2}{2g \sin \alpha}.$$

To find the time we have from (2)

$$v = v_0 - g \sin \alpha \cdot t;$$

and the particle stops when $v = 0$, in which case we have

$$t = \frac{v_0}{g \sin \alpha}.$$

From (6) we derive the following curious and useful result.

145. The Times of Descent down all Chords drawn from the Highest Point of a Vertical Circle are equal.—Let AB be the vertical diameter of the circle, AC any cord through A, α its inclination to the horizon; join BC; then if t be the time of descent down AC we have from (6) of Art. 144

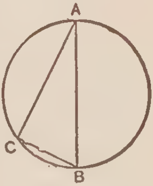

Fig.75

$$AC = \tfrac{1}{2} g t^2 \sin \alpha.$$

But $$AC = AB \sin \alpha;$$

$$\therefore \quad AB = \tfrac{1}{2}gt^2,$$

or
$$t = \sqrt{\frac{2AB}{g}},$$

which is constant, and shows that the time of falling down any chord is the same as the time of falling down the diameter.

Cor.—Similarly it may be shown that the times of descent down all chords drawn to B, the lowest point, are equal; that is, the time down CB is equal to that down AB.

146. The Straight Line of Quickest Descent from (1) a Given Point to a Given Straight Line (2) from a Given Point to a Given Curve.

(1) Let A be the given point and BC the given line. Through A draw the horizontal line AC, meeting CB in C; bisect the angle ACB by CO which intersects in O the vertical line drawn through A; from O draw OP perpendicular to BC; join AP; AP is the required line of quickest descent.

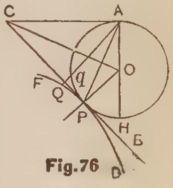

Fig.76

For OP is evidently equal to OA, and therefore the circle described with O as centre and with OP (= OA) for radius, will touch the line BC at P, and since the time of falling down all chords of this circle from A is the same, AP must be the line of quickest descent.

(2) To find the straight line of quickest descent to a given *curve*, all that is required is to draw a circle having the given point as the upper extremity of its vertical diameter, and tangent to the curve. Hence if DE (Fig. 76) be the curve, A the point, draw AH vertical; and, with centre in AH, describe a circle passing through A, and

touching DE at P, then AP is the required line. For, if we take any other point, Q, in DE, and draw AQ cutting the circle in q, then the time down AP = time down $Aq <$ time down AQ. Hence AP is the line of quickest descent.

The problem of finding the line of quickest descent from a point to a line or curve is thus found to resolve itself into the purely geometric problem of drawing a circle, the highest point of which shall be the given point and which shall touch the given line or curve.

EXAMPLES.*

1. If the earth travels in its orbit 600 million miles in $365\frac{1}{4}$ days, with uniform motion, what is its velocity in miles per second? *Ans.* 19·01 miles.

2. A train of cars moving with a velocity of 20 miles an hour, had been gone 3 hours when a locomotive was dispatched in pursuit, with a velocity of 25 miles an hour; in what time did the latter overtake the former? *Ans.* 12 hours.

3. A body moving from rest with a uniform acceleration describes 90 ft. in the 5th second of its motion; find the acceleration, f, and velocity, v, after 10 seconds. *Ans.* $f = 20$; $v = 200$.

4. Find the velocity of a particle which, moving with an acceleration of 20 ft. per sec. has traversed 1000 ft. *Ans.* 200 ft. per sec.

5. A body is observed to move over 45 ft. and 55 ft. in two successive seconds; find the space it would describe in the 20th second. *Ans.* 195 ft.

6. The velocity of a body increases every hour at the rate of 360 yards per hour. What is the acceleration, f, in feet per second, and what is the space, s, described from rest in 20 seconds? *Ans.* $f = 0.3$; $s = 60$ ft.

---

* In these examples take $g = 32$ ft.

7. A body is moving, at a given instant, at the rate of 8 ft. per sec.; at the end of 5 secs. its velocity is 19 ft. per sec. Assuming its acceleration to be uniform, what was its velocity at the end of 4 secs., and what will be its velocity at the end of 10 secs.? *Ans.* 16·8; 30.

8. A body is moving at a given instant with a velocity of 30 miles an hour, and comes to rest in 11 secs.; if the retardation is uniform what was its velocity 5 secs. before it stopped? *Ans.* 20 ft. per sec.

9. A body moves at the rate of 12 ft. a sec. with a uniform acceleration of 4; (1) state exactly what is meant by the number 4; (2) suppose the acceleration to go on for 5 secs., and then to cease, what distance will the body describe between the ends of the 5th and 12th secs.?

Ans. 224 ft.

10. A body, whose velocity undergoes a uniform retardation of 8, describes in 2 secs. a distance of 30 ft.; (1) what was its initial velocity? (2) How much longer than the 2 secs. would it move before coming to rest?

Ans. (1) 23; (2) $\frac{7}{8}$ sec.

11. A body whose motion is uniformly retarded, changes its velocity from 24 to 6 while describing a distance of 12 ft.; in what time does it describe the 12 ft.?

Ans. 0·8 sec.

12. The velocity of a body, which is at first 6 ft. a sec., undergoes a uniform acceleration of 3; at the end of 4 secs. the acceleration ceases; how far does the body move in 10 secs. from the beginning of the motion? *Ans.* 156 ft.

13. A body moves for a quarter of an hour with a uniform acceleration; in the first 5 minutes it describes 350 yards; in the second 5 minutes 420 yards; what is the whole distance described in a quarter of an hour?

Ans. 1260 yds.

14. Two secs. after a body is let fall another body is projected vertically downwards with a velocity of 100 ft. per sec.; when will it overtake the former?

Ans. $1\frac{7}{9}$ secs.

15. A body is projected upwards with a velocity of 100 ft. per sec.; find the whole time of flight. *Ans.* $6\frac{1}{4}$ secs.

16. A balloon is rising uniformly with a velocity of 10 ft. per sec., when a man drops from it a stone which reaches the ground in 3 secs.; find the height of the balloon, (1) when the stone was dropped; and (2) when it reached the ground. *Ans.* (1) 114 ft.; (2) 144 ft.

17. A man is standing on a platform which descends with a uniform acceleration of 5 ft. per sec.; after having descended for 2 secs. he drops a ball; what will be the velocity of the ball after 2 more seconds? *Ans.* 74 ft.

18. A balloon has been ascending vertically at a uniform rate for 4·5 secs., and a stone let fall from it reaches the ground in 7 secs.; find the velocity, v, of the balloon and the height, s, from which the stone is let fall.

Ans. $v = 174\frac{2}{7}$ ft. per sec.; $s = 784$ ft. If the balloon is still ascending when the stone is let fall $v = 68 \cdot 17$ ft. per sec.; $s = 306 \cdot 76$ ft.?

19. With what velocity must a particle be projected downwards, that it may in t secs. overtake another particle which has already fallen through a ft.?

$$Ans.\ v = \frac{a}{t} + \sqrt{2ag}.$$

20. A person while ascending in a balloon with a vertical velocity of V ft. per sec., lets fall a stone when he is h ft. above the ground; required the time in which the stone will reach the ground.

$$Ans.\ \frac{V + \sqrt{V^2 + 2gh}}{g}.$$

21. A body, A, is projected vertically downwards from the top of a tower with the velocity V, and one sec. afterwards another body, B, is let fall from a window a ft. from the top of the tower; in what time, t, will A overtake B?

$$Ans.\ t = \frac{2a + g}{2(V + g)}.$$

22. A stone let fall into a well, is heard to strike the bottom in t seconds; required the depth of the well, supposing the velocity of sound to be a ft. per sec.

$$Ans.\ \left[\sqrt{at + \frac{a^2}{2g}} - \frac{a}{\sqrt{2g}}\right]^2.$$

23. A stone is dropped into a well, and after 3 secs. the sound of the splash is heard. Find the depth to the surface of the water, the velocity of sound being 1127 ft. per sec. *Ans.* 132.9.

24. A body is simultaneously impressed with three uniform velocities, one of which would cause it to move 10 ft. North in 2 secs.; another 12 ft. in one sec. in the same direction; and a third 21 ft. South in 3 secs. Where will the body be in 5 secs.? *Ans.* 50 ft. North.

25. A boat is rowed across a river $1\frac{1}{4}$ miles wide, in a direction making an angle of 87° with the bank. The boat travels at the rate of 5 miles an hour, and the river runs at the rate of 2.3 miles an hour. Find at what point of the opposite bank the boat will land, if the angle of 87° be made against the stream.

Ans. 898 yards down the stream from the opposite point.

26. A body moves with a velocity of 10 ft. per sec. in a given direction; find the velocity in a direction inclined at an angle of 30° to the original direction.

Ans. $5\sqrt{3}$ ft. per sec.

27. A smooth plane is inclined at an angle of 30° to the horizon; a body is started up the plane with the velocity $5g$; find when it is distant $9g$ from the starting point.

Ans. 2, or 18 secs.

28. The angle of a plane is 30°; find the velocity with which a body must be projected up it to reach the top, the length of the plane being 20 ft.

Ans. $8\sqrt{10}$ ft. per sec.

29. A body is projected down a plane, the inclination of which is 45°, with a velocity of 10 ft.; find the space described in $2\frac{1}{2}$ secs. *Ans.* 95.7 ft. nearly.

30. A steam-engine starts on a downward incline of 1 in 200* with a velocity of $7\frac{1}{2}$ miles an hour neglecting friction; find the space traversed in two minutes.

Ans. 824 yards.

31. A body projected up an incline of 1 in 100 with a velocity of 15 miles an hour just reaches the summit; find the time occupied. *Ans.* 68.75 secs.

32. From a point in an inclined plane a body is made to slide up the plane with a velocity of 16.1 ft. per sec. (1) How far will it go before it comes to rest, the inclination of the plane to the horizon being 30°? (2) Also how far will the body be from the starting point after 5 secs. from the beginning of motion?

Ans. (1) 8.05 ft.; (2) 120.75 ft. lower down.

33. The inclination of a plane is 3 vertical to 4 horizontal; a body is made to slide up the incline with an initial velocity of 36 ft. a sec.; (1) how far will it go before beginning to return, and (2) after how many seconds will it return to its starting point?

Ans. (1) $33\frac{3}{4}$ ft.; (2) $3\frac{3}{4}$ secs.

---

* An incline of 1 in 200 means here 1 foot vertically to a *length* of 200 ft., though it is used by Engineers to mean 1 foot vertically to 200 ft. *horizontally*.

34. There is an inclined plane of 5 vertical to 12 horizontal, a body slides down 52 ft. of its length, and then passes without loss of velocity on to the horizontal plane; after how long from the beginning of the motion will it be at a distance of 100 ft. from the foot of the incline?

Ans. 5.7 secs.

35. A body is projected up an inclined plane, whose length is 10 times its height, with a velocity of 30 ft. per sec.; in what time will its velocity be destroyed?

Ans. $9\frac{3}{8}$ secs., if $g = 32$.

36. A body falls from rest down a given inclined plane; compare the times of describing the first and last halves of it.

Ans. As $1 : \sqrt{2} - 1$.

37. Two bodies, projected down two planes inclined to the horizon at angles of 45° and 60°, describe in the same time spaces respectively as $\sqrt{2} : \sqrt{3}$; find the ratio of the initial velocities of the projected bodies.

Ans. $\sqrt{2} : \sqrt{3}$.

38. Through what chord of a circle must a body fall to acquire half the velocity gained by falling through the diameter?

Ans. The chord which is inclined at 60° to the vertical.

39. Find the velocity with which a body should be projected down an inclined plane, l, so that the time of running down the plane shall be equal to the time of falling down the height, h.

Ans. $v = g\left(\frac{l - h \sin \alpha}{\sqrt{2gh}}\right)$.

40. Find the inclination of this plane, when a velocity of $\frac{3}{4}$ths that due to the height is sufficient to render the times of running down the plane, and of falling down the height, equal to each other.

Ans. 30°.

41. Through what chord of a circle, drawn from the bottom of the vertical diameter must a body descend, so as to acquire a velocity equal to $\frac{1}{n}$th part of the velocity acquired in falling down the vertical diameter?

Ans. If θ denote the angle between the required chord and the vertical diameter $\cos\theta = \frac{1}{n}$.

42. Find the inclination, θ, of the radius of a circle to the vertical, such that a body running down will describe the radius in the same time that another body requires to fall down the vertical diameter. *Ans.* $\theta = 60°$.

43. Find the inclination, θ, to the vertical of the diameter down which a body falling will describe the last half in the same time as the vertical diameter.

$$Ans.\ \cos\theta = \frac{3\sqrt{2} - 4}{2\sqrt{2}}.$$

44. Show that the times of descent down all the radii of curvature of the cycloid (Fig. 40, Calculus) are equal; that is, the time down PQ is equal to the time down $O'A = \sqrt{\frac{8r}{g}}$

45. Find the inclination, θ, to the horizon of an inclined plane, so that the time of descent of a particle down the length may be n times that down the height of the plane.

$$Ans.\ \theta = \sin^{-1}\frac{1}{n}.$$

46. Find the line of quickest descent from the focus to a parabola whose axis is vertical and vertex upwards, and show that its length is equal to that of the latus rectum.

47. Find the line of quickest descent from the focus of a parabola to the curve when the axis is horizontal.

48. Find geometrically the line of quickest descent (1) from a point within a circle to the circle; (2) from a circle to a point without it.

49. Find geometrically the straight line of longest descent from a circle to a point without it, and which lies below the circle.

50. A man six feet high walks in a straight line at the rate of four miles an hour away from a street lamp, the height of which is 10 feet; supposing the man to start from the lamp-post, find the rate at which the end of his shadow travels, and also the rate at which the end of his shadow separates from himself.

Ans. Shadow travels 10 miles an hour, and gains on himself 6 miles an hour.

51. Two bodies fall in the same time from two given points in space in the same vertical down two straight lines drawn to any point of a surface; show that the surface is an equilateral hyperboloid of revolution, having the given points as vertices.

52. Find the form of a curve in a vertical plane, such that if heavy particles be simultaneously let fall from each point of it so as to slide freely along the normal at that point, they may all reach a given horizontal straight line at the same instant.

53. Show that the time of quickest descent down a focal chord of a parabola whose axis is vertical is

$$\sqrt{\frac{3^{\frac{3}{2}}l}{g}},$$

where l is the latus rectum.

54. Particles slide from rest at the highest point of a vertical circle down chords, and are then allowed to move

freely; show that the locus of the foci of their paths is a circle of half the radius, and that all the paths bisect the vertical radius.

55. If the particles slide down chords to the lowest point, and be then suffered to move freely, the locus of the foci is a cardioid.

56. Particles fall down *diameters* of a vertical circle; the locus of the foci of their subsequent paths is the circle.

CHAPTER II.

CURVILINEAR MOTION.

147. Remarks on Curvilinear Motion.—The motion, which was considered in the last chapter, was that of a particle describing a rectilinear path. In this chapter the circumstances of motion in which the path is *curvilinear* will be considered. The conception and the definition of velocity and of acceleration which were given in Arts. 134, 135, are evidently as applicable to a particle describing a curvilinear path as to one moving along a straight line; and consequently the formulæ for velocity in Arts. 142, 143, are applicable either to rectilinear or to curvilinear motion. In the last chapter the effects of the composition and the resolution of velocities were considered, when the path taken by the particle in consequence of them was straight; we have now to investigate the effects of velocities and of accelerations in a more general way.

148. Composition of Uniform Velocity and Acceleration.—Suppose a body tends to move in one direction with a uniform velocity which would carry it from A to B in one second, and also subject to an acceleration that would carry it from A to C in one second; then at the end of the second the body will be at D, the opposite end of the diagonal of the parallelogram ABDC, just as if it had moved from A to B and then from B to D in the second, but the body will move in the *curve* and not along the *diagonal*. For, the body in its motion is making progress uniformly in the direction AB, at the same rate as if it had no other motion; and at the same time it is being accelerated in the

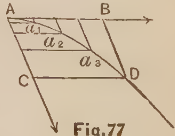

Fig. 77

direction AC, as fast as if it had no other motion. Hence the body will reach D as far from the line AC as if it had moved over AB, and as far from AB as if it had moved over AC; but since the velocity along AC is not uniform, the spaces described in equal intervals of times will not be equal along AC while they are equal along AB, and therefore the points a_1, a_2, a_3, will not be in a straight line. In this case, therefore, the path is a curve.

149. Composition and Resolution of Accelerations.—If a body is subject to two different accelerations in different directions the sides of a parallelogram may be taken to represent the *Component Accelerations*, and the diagonal will represent the *Resultant Acceleration*, although the path of the body may be along some other line.

REM.—These results with those of Arts. 142, 143, may be summed up in one general law: *When a body tends to move with several different velocities in different directions, the body will be, at the end of any given time, at the same point, as if it had moved with each velocity separately.* This is the fundamental law of the composition of velocities, and it shows that all problems which involve tendencies to motion in different directions *simultaneously*, may be treated as if those tendencies were *successive*.*

If $\frac{d^2s}{dt^2}$ be the acceleration along the curve, and (x, y, z) be the place of the moving particle at the time, t, it is evident that the component accelerations parallel to the axes are $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$. Denoting these by α_x, α_y, α_z, we have

$$\frac{d^2x}{dt^2} = \alpha_x; \quad \frac{d^2y}{dt^2} = \alpha_y; \quad \frac{d^2z}{dt^2} = \alpha_z;$$

and $\sqrt{\alpha_x^2 + \alpha_y^2 + \alpha_z^2}$ is the *resultant acceleration*.

---

* See Remarks on Newton's 2d law, Art. 168.

Also if α, β, γ, be the angles which the direction of motion makes with the axes, we have

$$\frac{d^2x}{dt^2} = \frac{d^2s}{dt^2}\cos\alpha = \alpha_x;$$

$$\frac{d^2y}{dt^2} = \frac{d^2s}{dt^2}\cos\beta = \alpha_y;$$

$$\frac{d^2z}{dt^2} = \frac{d^2s}{dt^2}\cos\gamma = \alpha_z.$$

The acceleration $\frac{d^2s}{dt^2}$ is not generally the complete resultant of the three component accelerations, but is so only when the path is a straight line or the velocity is zero. It is, however, the only part of their resultant which has any effect on the velocity. $\frac{d^2s}{dt^2}$ is the sum of the resolved parts of the component accelerations in the direction of motion, as the following identical equation shows:

$$\frac{d^2s}{dt^2} = \frac{dx}{ds}\cdot\frac{d^2x}{dt^2} + \frac{dy}{ds}\cdot\frac{d^2y}{dt^2} + \frac{dz}{ds}\cdot\frac{d^2z}{dt^2},$$

which follows immediately from (1) of Art. 143 by differentiation. Accelerations are therefore subject to the same laws of composition and resolution as velocities; and consequently the acceleration of the particle along any line is the sum of the resolved parts of the axial accelerations along that line. Thus to find $\frac{d^2s}{dt^2}$, the acceleration along s, $\frac{d^2x}{dt^2}$ has to be multiplied by $\frac{dx}{ds}$, which is the direction-cosine of the small arc ds. The other part of the resultant is at right angles to this, and its only effect is to change the *direction* of the motion of the point. (See Tait and Steele's Dynamics of a Particle, also Thomson and Tait's Nat. Phil.)

The following are examples in which the preceding expressions are applied to cases in which the laws of velocity and of acceleration are given.

EXAMPLES.

1. A particle moves so that the axial components of its velocity vary as the corresponding co-ordinates; it is required to find the equation of its path; and the accelerations along the axes.

Here $$\frac{dx}{dt} = kx; \quad \frac{dy}{dt} = ky;$$

$$\therefore \quad \frac{dx}{x} = \frac{dy}{y} = k\,dt;$$

$$\therefore \quad \log\frac{x}{a} = \log\frac{y}{b} = kt,$$

if (a, b) is the initial place of the particle,

$$\therefore \quad x = ae^{kt}; \quad y = be^{kt}.$$

$$\therefore \quad \frac{x}{a} = \frac{y}{b}$$

is the equation of the path.

And the axial accelerations are

$$\frac{d^2x}{dt^2} = k^2x; \quad \frac{d^2y}{dt^2} = k^2y.$$

2. A wheel rolls along a straight line with a uniform velocity; compare the velocity of a given point in the circumference with that of the centre of the wheel.

Let the line along which the wheel rolls be the axis of x, and let v be the velocity of its centre; then a point in its circumference describes a cycloid, of which, the origin being taken at its starting point, the equation is

$$x = a \operatorname{vers}^{-1}\frac{y}{a} - (2ay - y^2)^{\frac{1}{2}};$$

$$\therefore \quad \frac{dx}{y^{\frac{1}{2}}} = \frac{dy}{(2a-y)^{\frac{1}{2}}} = \frac{ds}{(2a)^{\frac{1}{2}}}.$$

But $\quad v = \dfrac{d}{dt}\left(a \operatorname{vers}^{-1}\dfrac{y}{a}\right) = \dfrac{a}{(2ay-y^2)^{\frac{1}{2}}} \cdot \dfrac{dy}{dt};$

$$\therefore \quad \frac{ds}{dt} = \frac{ds}{dy} \cdot \frac{dy}{dt} = \left(\frac{2y}{a}\right)^{\frac{1}{2}} v;$$

which is the velocity of the point in the circumference of the wheel. Thus the velocity of the highest point of the wheel is twice as great as that of the centre, while the point that is in contact with the straight line has no velocity. (See Price's Anal. Mech's., Vol. I, p. 416.)

3. If $\dfrac{dx}{dt} = ky$, $\dfrac{dy}{dt} = kx$, show that the path is an equilateral hyperbola and that the axial components are

$$\frac{d^2x}{dt^2} = k^2x, \quad \frac{d^2y}{dt^2} = k^2y.$$

4. A particle describes an ellipse so that the x-component of its velocity is a constant, α; find the y-component of its velocity and acceleration, and the time of describing the ellipse.

Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

and let (x, y) be the position of the particle at the time t;

then $\qquad \dfrac{dx}{dt} = \alpha; \quad$ and $\quad \dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y};$

$$\therefore \quad \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = -\frac{\alpha b^2}{a^2} \cdot \frac{x}{y},$$

which is the y-component of the velocity.

Also
$$\frac{d^2y}{dt^2} = -\frac{\alpha b^2}{a^2}\cdot\frac{y\dfrac{dx}{dt} - x\dfrac{dy}{dt}}{y^2}$$
$$= -\frac{b^4\alpha^2}{a^2y^3};$$

hence the acceleration parallel to the axis of y varies inversely as the cube of the ordinate of the ellipse, and acts towards the axis of x, as is shown by the negative sign.

The time of passing from the extremity of the minor axis to that of the major axis is found by dividing a by α, the constant velocity parallel to the axis of x, giving $\frac{a}{\alpha}$, and the time of describing the whole ellipse is $\frac{4a}{\alpha}$.

If the orbit is a circle $b = a$, and the acceleration parallel to the axis of y is $-\frac{a^2\alpha^2}{y^3}$.

If the velocity parallel to the y-axis is constant and equal to β, then
$$\frac{dx}{dt} = -\frac{a^2\beta}{b^2}\cdot\frac{y}{x};$$
$$\frac{d^2x}{dt^2} = -\frac{a^4\beta^2}{b^2x^3};$$
and the periodic time $= \frac{4b}{\beta}$.

5. A particle describes the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; find (1) the acceleration parallel to the axis of x if the velocity parallel to the axis of y is a constant, β, and (2) find the acceleration parallel to the y-axis if the velocity parallel to the x-axis is a constant α.

(1) Here we have
$$\frac{dy}{dt} = \beta; \quad \text{and} \quad \frac{dy}{dx} = \frac{b^2}{a^2}\cdot\frac{x}{y};$$

$$\therefore \quad \frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt} = \frac{\beta a^2}{b^2} \cdot \frac{y}{x}$$

which is the velocity parallel to the x-axis.

Also
$$\frac{d^2x}{dt^2} = \frac{\beta a^2}{b^2} \cdot \frac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2}$$

$$= \frac{\beta^2 a^4}{b^2 x^3},$$

hence the acceleration parallel to the x-axis varies inversely as the cube of the abscissa, and the x-component of the velocity is increasing.

(2) Here we have
$$\frac{dx}{dt} = \alpha;$$

$$\therefore \quad \frac{dy}{dt} = \frac{\alpha b^2}{a^2} \cdot \frac{x}{y};$$

and
$$\frac{d^2y}{dt^2} = -\frac{\alpha^2 b^4}{a^2 y^3};$$

hence the acceleration parallel to the y-axis is negative and the y-component of the velocity is decreasing.

6. A particle describes the parabola, $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, with a constant velocity, c; find the accelerations parallel to the axes of x and y.

Here we have
$$\frac{ds}{dt} = c;$$

and
$$\frac{dx}{x^{\frac{1}{2}}} = \frac{-dy}{y^{\frac{1}{2}}} = \frac{ds}{(x+y)^{\frac{1}{2}}};$$

$$\therefore \qquad \frac{dx^2}{dt^2} = \frac{ds^2}{dt^2} \cdot \frac{x}{x+y} = \frac{c^2x}{x+y};$$

and
$$\frac{dy^2}{dt^2} = \frac{ds^2}{dt^2} \cdot \frac{y}{x+y} = \frac{c^2y}{x+y};$$

differentiating we get

$$\frac{d^2x}{dt^2} = \frac{c^2\,(ay)^{\frac{1}{2}}}{2\,(x+y)^2};$$

$$\frac{d^2y}{dt^2} = \frac{c^2\,(ax)^{\frac{1}{2}}}{2\,(x+y)^2}.$$

7. A particle describes a parabola with such a varying velocity that its projection on a line perpendicular to the axis is a constant, v. Find the velocity and the acceleration parallel to the axis.

Let the equation of the parabola be

$$y^2 = 2px;$$

then
$$\frac{dy}{dt} = v,$$

and
$$\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt} = \frac{vy}{p};$$

which is the velocity parallel to x

Also
$$\frac{d^2x}{dt^2} = \frac{v^2}{p},$$

which shows that the particle is moving away from the tangent to the curve at the vertex with a constant acceleration.

Hence as the earth acts on particles near its surface with a constant acceleration in vertical lines, if a particle is projected with a velocity, v, in a horizontal line it will move in a parabolic path.

150. Motion of Projectiles in Vacuo.—If a particle be projected in a direction oblique to the horizon it is called a *Projectile,* and the path which it describes is called its *Trajectory.* The case which we shall here consider is that of a particle moving in vacuo under the action of gravity; so that the problem is that of the *motion of a projectile in vacuo;* and hence, as gravity does not affect its horizontal velocity, it resolves itself into the purely kinematic problem of a particle moving so that its horizontal acceleration is 0 and its vertical acceleration is the constant, g, (Art. 140).

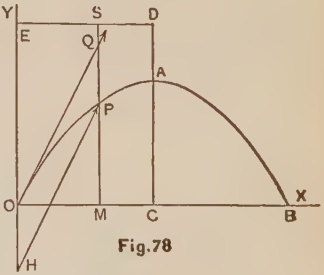

Fig.78

151. The Path of a Projectile in Vacuo is a Parabola.—Let the plane in which the particle is projected be the plane of xy; let the axis of x be horizontal and the axis of y vertical and positive upwards, the origin being at the point of projection; let the velocity of projection $= v$, and let the line of projection be inclined at an angle α to the axis of x, so that $v \cos \alpha$, and $v \sin \alpha$ are the resolved parts of the velocity of projection along the axes of x and y. It is evident that the particle will continue to move in the plane of xy, as it is projected in it, and is subject to no force which would tend to withdraw it from that plane.

Let (x, y) be the place, P, of the particle at the time t; then the equations of motion are

$$\frac{d^2x}{dt^2} = 0; \quad \frac{d^2y}{dt^2} = -g;$$

the acceleration being negative since the y-component of the velocity is decreasing.

The first and second integrals of these equations will then be, taking the limits corresponding to $t = t$ and $t = 0$,

$$\frac{dx}{dt} = v \cos \alpha; \quad \frac{dy}{dt} = v \sin \alpha - gt; \tag{1}$$

$$x = v \cos \alpha t; \quad y = v \sin \alpha t - \tfrac{1}{2}gt^2. \tag{2}$$

Equations (1) and (2) give the coordinates of the particle and its velocity parallel to either axis at any time, t.

Eliminating t between equations (2) we obtain

$$y = x \tan \alpha - \frac{gx^2}{2v^2 \cos^2 \alpha} \tag{3}$$

which is the equation of the trajectory, and shows that the particle will move in a parabola.

152. The Parameter; the Range R; the Greatest Height H; Height of the Directrix.

—Equation (3) of Art. 151 may be written

$$x^2 - \frac{2v^2 \sin \alpha \cos \alpha}{g} x = -\frac{2v^2 \cos^2 \alpha}{g} y,$$

or $$\left(x - \frac{v^2 \sin \alpha \cos \alpha}{g}\right)^2 = -\frac{2v^2 \cos^2 \alpha}{g}\left(y - \frac{v^2 \sin^2 \alpha}{2g}\right). \tag{1}$$

By comparing this with the equation of a parabola referred to its vertex as origin, we find for

$$\text{the abscissa of the vertex} = \frac{v^2 \sin \alpha \cos \alpha}{g}; \tag{2}$$

$$\text{the ordinate of the vertex} = \frac{v^2 \sin^2 \alpha}{2g}; \tag{3}$$

$$\textit{the parameter } (\text{latus rectum}) = -\frac{2v^2 \cos^2 \alpha}{g}. \tag{4}$$

And by transferring the origin to the vertex (1) becomes

$$x^2 = -\frac{2v^2 \cos^2 \alpha}{g} y \tag{5}$$

which is the equation of a parabola with its axis vertical and the vertex the highest point of the curve.

The distance, OB, between the point of projection and the point where the projectile strikes the horizontal plane is called *the Range* on the horizontal plane, and is the value of x when $y = 0$. Putting $y = 0$ in (3) of Art. 151 and solving for x, we get

$$\textit{the horizontal range } R = \text{OB} = \frac{v^2 \sin 2\alpha}{g}; \tag{6}$$

which is evident, also, geometrically, as $\text{OB} = 2\text{OC}$; that is, the range is equal to twice the abscissa of the vertex.

It follows from (6) that the range is the greatest, for a given velocity of projection, when $\alpha = 45°$, in which case the range $= \frac{v^2}{g}$.

Also it appears from (6) that the range is the same when α is replaced by its complement; that is, for the same velocity of projection the range is the same for two different angles that are complements of each other. If $\alpha = 45°$ the two angles become identical, and the range is a maximum.

CA is called the *greatest height*, H, of the projectile, and is given by (3) which, when $\alpha = 45°$ becomes $\frac{v^2}{4g}$. (7)

The height of the directrix = CD

$$= \frac{v^2 \sin^2 \alpha}{2g} + \tfrac{1}{4}\frac{2v^2 \cos^2 \alpha}{g} = \frac{v^2}{2g}. \qquad (8)$$

Hence when $\alpha = 45°$ the focus of the parabola lies in the horizontal line through the point of projection.

153. The Velocity of the Particle at any Point of its Path.—Let V be the velocity at any point of its path,

then $\quad V^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$, or by (1) of Art. 151

$$= v^2 \cos^2 \alpha + (v^2 \sin^2 \alpha - 2v \sin \alpha gt + g^2t^2)$$

$$= v^2 - 2gy.$$

To acquire this velocity in falling from rest, the particle must have fallen through a height $\frac{V^2}{2g}$, (6) of Art. 140, or its equal

$$\frac{v^2}{2g} - y = MS - MP \text{ by } (8)$$

$$= PS.$$

Hence, the velocity at any point, P, on the curve is that which the particle would acquire in falling freely in vacuo down the vertical height SP; that is, in falling from the directrix to the curve; and the velocity of projection at O is that which the particle would acquire in falling freely through the height CD. The directrix of the parabola is therefore determined by the velocity of projection, and is at a vertical distance above the point of projection equal to that down which a particle falling would have the velocity of projection.

154. The Time of Flight, *T*, along a Horizontal Plane.—Put $y = 0$ in (3) of Art. 151, and solve for x, the values of which are 0 and $\frac{2v^2 \sin \alpha \cos \alpha}{g}$. But the horizontal velocity is $v \cos \alpha$. Hence *the time of flight* $= \frac{2v \sin \alpha}{g}$ which varies as the sine of the inclination to the axis of x.

155. To Find the Point at which a Projectile will Strike a Given Inclined Plane passing through the Point of Projection, and the Time of Flight.—Let the inclined plane make an angle β with the horizon; it is evident that we have only to eliminate y between $y = x \tan \beta$ and (3) of Art. 151, which gives for the abscissa of the point where the projectile meets the plane

$$x_1 = \frac{2v^2 \cos \alpha \sin (\alpha - \beta)}{g \cos \beta};$$

and the ordinate is

$$y_1 = \frac{2v^2 \cos \alpha \tan \beta \sin (\alpha - \beta)}{g \cos \beta}. \qquad (1)$$

Hence the time of flight

$$T = \frac{x_1}{v \cos \alpha} = \frac{2v \sin (\alpha - \beta)}{g \cos \beta}. \qquad (2)$$

156. The Direction of Projection which gives the Greatest Range on a Given Plane.—The range on the horizontal plane is

$$\frac{v^2 \sin 2\alpha}{g},$$

which for a given value of v is greatest when $\alpha = \frac{\pi}{4}$ (Art. 152).

The range on the inclined plane $= x_1 \sec \beta$

$$= \frac{2v^2 \cos \alpha \sin (\alpha - \beta)}{g \cos^2 \beta}. \qquad (1)$$

To find the value of α which makes this a maximum, we must equate to zero its derivative with respect to α, which gives

$$\cos(2\alpha - \beta) = 0;$$

$$\therefore \quad \alpha = \tfrac{1}{2}\left(\frac{\pi}{2} + \beta\right); \tag{2}$$

and hence
$$\alpha - \beta = \tfrac{1}{2}\left(\frac{\pi}{2} - \beta\right), \tag{3}$$

which is the angle which the direction of projection makes with the inclined plane when the range is a maximum; that is, the projection bisects the angle between the inclined plane and the vertical.

In this case by substituting in (1) the values of α and $(\alpha - \beta)$ as given in (2) and (3) and reducing, we get

$$\text{the greatest range} = \frac{v^2}{g(1 + \sin\beta)}. \tag{4}$$

157. The angle of Elevation so that the Particle may pass through a Given Point.—From Art. 152, there are two directions in which a particle may be projected so as to reach a given point; and they are equally inclined to the direction of projection $\left(\alpha = \frac{\pi}{4}\right)$.

Let the given point lie in the plane which makes an angle β with the horizon, and suppose its abscissa to be h; then we must have from (1) of Art. 155

$$\frac{2v^2}{g\cos\beta}\cos\alpha\sin(\alpha - \beta) = h.$$

If α' and α'' be the two values of α which satisfy this equation, we must have

$$\cos\alpha'\sin(\alpha' - \beta) = \cos\alpha''\sin(\alpha'' - \beta);$$

and therefore $$\alpha'' - \beta = \frac{\pi}{2} - \alpha',$$

or $$\alpha'' - \tfrac{1}{2}\left(\frac{\pi}{2} + \beta\right) = \tfrac{1}{2}\left(\frac{\pi}{2} + \beta\right) - \alpha'. \qquad (1)$$

But each member of (1) is the angle between one of the directions of projection and the direction for the greatest range [Art. 156, (2)]. Hence, as in Art. 152, the two directions of projection which enable the particle to pass through a point in a given plane through the point of projection, are equally inclined to the direction of projection for the greatest range along that plane. (See Tait and Steele's Dynamics of a Particle, p. 89.)

158. Second Method of Finding the Equation of the Trajectory.—By a somewhat simpler method than that of Art. 151, we may find the equation of the path of the projectile as the resultant of a uniform velocity and an acceleration (Art. 148).

Take the direction of projection (Fig. 78) as the axis of x, and the vertical downwards from the point of projection as the axis of y. Then (Art. 149, Rem.) the velocity, v, due to the projection, will carry the particle, with uniform motion, parallel to the axis of x, while at the same time, it is carried with constant acceleration, g, parallel to the axis of y. Hence at any time, t, the equations of motion along the axes of x and y respectively are

$$x = vt,$$

$$y = \tfrac{1}{2}gt^2.$$

That is, if the particle were moving with the velocity v, alone, it would in the time t, arrive at Q; and if it were then to move with the vertical acceleration g alone it would in the same time arrive at P; therefore if the velocity v

and the acceleration g are *simultaneous*, the particle will in the time t arrive at P (Art. 149, Rem).

Eliminating t we have

$$x^2 = \frac{2v^2}{g} y,$$

which is the equation of a parabola referred to a diameter and the tangent at its vertex. The distance of the origin from the directrix, being $\frac{1}{4}$th of the coefficient of y, is $\frac{v^2}{2g}$, as in Art. 152, (8).

EXAMPLES.

1. From the top of a tower two particles are projected at angles α and β to the horizon with the same velocity, v, and both strike the horizontal plane passing through the bottom of the tower at the same point; find the height of the tower.

Let h = the height of the tower; v = the velocity of projection; then if the particles are projected from the edge of the top of the tower, and x is the distance from the bottom of the tower to the point where they strike the horizontal plane we have from (3) of Art. 151

$$-h = x \tan \alpha - \frac{gx^2}{2v^2}(1 + \tan^2 \alpha), \tag{1}$$

$$-h = x \tan \beta - \frac{gx^2}{2v^2}(1 + \tan^2 \beta), \tag{2}$$

by subtraction

$$x = \frac{2v^2}{g(\tan \alpha + \tan \beta)} = \frac{2v^2 \cos \alpha \cos \beta}{g \sin(\alpha + \beta)};$$

which in (1) or (2) gives

$$h = \frac{2v^2 \cos \alpha \cos \beta \cos(\alpha + \beta)}{g[\sin(\alpha + \beta)]^2}.$$

2. Particles are projected with a given velocity in all lines in a vertical plane from the point O; it is required to find the locus of their highest points.

Let (x, y) be the highest point; then from (2) and (3) of Art. 152, we have

$$x = \frac{v^2 \sin \alpha \cos \alpha}{g};$$

$$y = \frac{v^2 \sin^2 \alpha}{2g};$$

therefore $\sin^2 \alpha = \frac{2gy}{v^2}$, and $\cos^2 \alpha = \frac{gx^2}{2v^2y}$.

Adding $$4y^2 + x^2 = \frac{2v^2y}{g};$$

which is the equation of an ellipse, whose major axis $= \frac{v^2}{g}$; and the minor axis $= \frac{v^2}{2g}$; and the origin is at the extremity of the minor axis.

3. Find the angle of projection, α, so that the area contained between the path of the projectile and the horizontal line may be a maximum, and find the value of the maximum area.

Ans. $\alpha = 60°$ and Max. Area $= \frac{v^4}{8g^2}(3)^{\frac{1}{2}}$.

4. Find the ratio of the areas A_1 and A_2 of the two parabolas described by projectiles whose horizontal ranges are the same, and the angles of projection are therefore complements of each other.

Ans. $\frac{A_1}{A_2} = \tan^2 \alpha$.

159. Velocity of Discharge of Balls and Shells from the Mouth of a Gun.—As the result of numerous experiments made at Woolwich, the following formula was regarded as a correct expression for the velocity of balls and shells, on quitting the gun, and fired with moderate charges of powder, from the pieces of ordnance commonly used for military purposes:

$$v = 1600\sqrt{\frac{3P}{W}},$$

where v is the velocity in feet per second, P the weight of the charge of powder, and W the weight of the ball.

For the investigation of the path of a projectile in the atmosphere, see Chap. I of Kinetics.

160. Angular Velocity, and Angular Acceleration.—Hitherto the method of resolving velocities and accelerations along two rectangular axes has been employed. It remains for us to investigate the kinematics of a particle describing a curvilinear path, from another point of view and in relation to another system of reference. Before we consider velocities and accelerations in reference to a system of polar co-ordinates, it is necessary to enquire into a mode of measuring the *angular velocity* of a particle.

Angular Velocity may be defined as the rate of angular motion. Thus let (r, θ) be the position of the point P, and suppose that the radius vector has revolved uniformly through the angle θ in the time t, then denoting the angular velocity by ω, we shall have, as in linear velocity (Art. 7)

$$\omega = \frac{\theta}{t}.$$

If however the radius vector does not revolve uniformly through the angle θ we may always regard it as revolving uniformly through the angle $d\theta$ in the infinitesimal of time dt; hence we shall have as the proper value of ω,

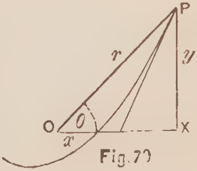

Fig. 79

$$\omega = \frac{d\theta}{dt}. \qquad (1)$$

Hence, whether the angular velocity be uniform or variable, it is the ratio of the angle described by the radius vector in a given time to the time in which it is described; thus the increase of the angle, in angular velocity, takes the place of the increase of the distance from a fixed point, in linear velocity, (Art. 7).

Angular Acceleration is the rate of increase of angular velocity; it is a velocity increment, and is measured in the same way as *linear acceleration* (Art. 9). Thus, whether the angular acceleration is uniform or variable, it may always be regarded as uniform during the infinitesimal of time dt in which time the increment of the velocity will be $d\omega$. Hence denoting the angular acceleration at any time, t, by ϕ, we have

$$\phi = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) \text{ from (1)}$$

$$= \frac{d^2\theta}{dt^2}, \qquad (2)$$

and thus, whether the increase of angular velocity is uniform or variable, the angular acceleration is the increase of angular velocity in a unit of time.

The following examples are illustrations of the preceding mode of estimating velocities and accelerations.

EXAMPLES.

1. If a particle is placed on the revolving line at the distance r from the origin, and the line revolves with a uniform angular velocity, ω, the relation between the linear velocity of the particle and the angular velocity may thus be found.

Let $d\theta$ be the angle through which the radius revolves in the time dt, and let ds be the path described by the particle, so that $ds = rd\theta$;

then $$\frac{ds}{dt} = r\frac{d\theta}{dt} = \omega r;$$

so that the linear velocity varies as the angular velocity and the length of the radius jointly.

2. If the angular acceleration is a constant, as ϕ; then from (2) we have

$$\frac{d^2\theta}{dt^2} = \phi;$$

$$\therefore \quad \frac{d\theta}{dt} = \phi t + \omega_0,$$

and $$\therefore \quad \theta = \tfrac{1}{2}\phi t^2 + \omega_0 t + \theta_0,$$

where ω_0 and θ_0 are the initial values of ω and θ.

Hence if a line revolves from rest with a constant angular acceleration, we have

$$\theta = \tfrac{1}{2}\phi t^2;$$

and the angle described by it varies as the square of the time.

3. If a particle revolves in a circle uniformly, and its place is continually projected on a given diameter, the linear acceleration along that diameter varies directly as the distance of the projected place from the centre.

Let ω be the constant angular velocity, θ the angle between the fixed diameter and the radius drawn from the centre to its place at the time t, x the distance of this projected place from the centre. Then, calling a the radius of the circle, we have

$$x = a\cos\theta,$$

$$\frac{dx}{dt} = -a \sin \theta \frac{d\theta}{dt} = -a\omega \sin \theta;$$

$$\frac{d^2x}{dt^2} = -a\omega \cos \theta \frac{d\theta}{dt} = -\omega^2 x;$$

which proves the theorem.

4. If the angular acceleration varies as the angle generated from a given fixed line, and is negative, find the angle.

Here the equation which expresses the motion is of the form

$$\frac{d^2\theta}{dt^2} = -k^2\theta.$$

Calling α the initial value of θ we find for the result

$$\theta = \alpha \cos kt.$$

5. If a particle revolves in a circle with a uniform velocity, show that its angular velocity about any point in the circumference is also uniform, and equal to one-half of what it is about the centre.

At present this is sufficient for the general explanation of angular velocity and angular acceleration. We shall return to the subject in Chap. 7, Part III., when we treat of the motion of rigid bodies.

161. The Component Accelerations, at any instant, Along, and Perpendicular to the Radius Vector.—Let (r, θ) (Fig. 79) be the place of the moving particle, P, at the time t, (x, y) being its place referred to a system of rectangular axes having the same origin, and the x-axis coincident with the initial line. Then

$$x = r \cos \theta; \quad y = r \sin \theta; \tag{1}$$

therefore $$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}; \qquad (2)$$

and

$$\frac{d^2x}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\cos\theta - \left[2\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\sin\theta. \quad (3)$$

Similarly

$$\frac{d^2y}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\sin\theta + \left[2\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\cos\theta; \quad (4)$$

which are the accelerations parallel to the axes of x and y. Resolving these along the radius vector by multiplying (3) and (4) by $\cos\theta$ and $\sin\theta$ respectively, since accelerations may be resolved and compounded along any line the same as velocities (Art. 149), and adding, we have

$$\frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2; \qquad (5)$$

which *is the acceleration along the radius vector.*[*]

Multiplying (3) and (4) by $\sin\theta$ and $\cos\theta$ respectively, and subtracting the former from the latter, we get

$$\begin{aligned}\frac{d^2y}{dt^2}\cos\theta - \frac{d^2x}{dt^2}\sin\theta &= 2\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\\ &= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right); \qquad (6)\end{aligned}$$

which *is the acceleration perpendicular to the radius vector.*[†]

162. The Component Accelerations, at any instant, Along, and Perpendicular to the Tangent.—Let (x, y) (Fig. 79) be the place of the moving particle, P, at the time t, and s the length of the arc described during that time. Then the accelerations along the axes of x and y are $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$; and the *direction cosines** are $\frac{dx}{ds}$ and $\frac{dy}{ds}$. To find the acceleration along the tangent we must multiply these axial accelerations by $\frac{dx}{ds}$ and $\frac{dy}{ds}$, respectively, and add. Thus the tangential acceleration, T, is

---

[*] Sometimes called *the radial acceleration.*

[†] Sometimes called *the transversal acceleration.*

$$T = \frac{d^2x}{dt^2} \cdot \frac{dx}{ds} + \frac{d^2y}{dt^2} \cdot \frac{dy}{ds}. \tag{1}$$

Since $ds^2 = dx^2 + dy^2$, therefore, by differentiation we have

$$ds\, d^2s = dx\, d^2x + dy\, d^2y;$$

and dividing by $ds\, dt^2$ we get

$$\frac{d^2s}{dt^2} = \frac{d^2x}{dt^2} \cdot \frac{dx}{ds} + \frac{d^2y}{dt^2} \cdot \frac{dy}{ds},$$

which in (1) gives

$$T = \frac{d^2s}{dt^2}, \tag{2}$$

for *the acceleration along the tangent.*

Similarly we have for the normal acceleration, **N,**

$$\begin{aligned} N &= \frac{d^2y}{dt^2} \cdot \frac{dx}{ds} - \frac{d^2x}{dt^2} \cdot \frac{dy}{ds} \\ &= \frac{(d^2y\, dx - d^2x\, dy)}{ds^3} \cdot \frac{ds^2}{dt^2} \\ &= \frac{1}{\rho} \cdot \frac{ds^2}{dt^2}, \text{ (by Ex. 4, p. 144, Calculus),} \end{aligned}$$

where ρ is the radius of curvature;

---

* Cosines of the angles which the tangent makes with the axes of x and y.

$$\therefore \quad \mathrm{N} = \frac{v^2}{\rho}, \tag{3}$$

if v is the velocity of the particle at the point (x, y).

Hence at any point, P, of the trajectory, if the acceleration is resolved along the tangent to the curve at P and along the normal, the accelerations along the two lines are respectively

$$\frac{d^2s}{dt^2} \quad \text{and} \quad \frac{v^2}{\rho}.$$

163. When the Acceleration Perpendicular to the Radius Vector is zero.—Then from (6) of Art. 161 we have

$$r^2\frac{d\theta}{dt} = \text{constant} = h \text{ suppose};$$

$$\therefore \quad \frac{d\theta}{dt} = \frac{h}{r^2};$$

and

$$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = \frac{h}{r^2}\cdot\frac{dr}{d\theta};$$

$$\because \quad \frac{d^2r}{dt^2} = \frac{h^2}{r^4}\cdot\frac{d^2r}{d\theta^2} - 2\frac{h^2}{r^5}\left(\frac{dr}{d\theta}\right)^2;$$

which in (5) of Art. 161 gives

the acceleration along the radius vector

$$= \frac{h^2}{r^4}\frac{d^2r}{d\theta^2} - 2\frac{h^2}{r^5}\left(\frac{dr}{d\theta}\right)^2 - \frac{h^2}{r^3}; \tag{1}$$

an expression which is independent of t.

This may be put into a more convenient form as follows:

Let $r = \frac{1}{u}$; then

$$\frac{dr}{d\theta} = -\frac{1}{u^2}\cdot\frac{du}{d\theta};$$

$$\therefore \quad \frac{d^2r}{d\theta^2} = -\frac{1}{u^2}\cdot\frac{d^2u}{d\theta^2} + \frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2;$$

which in (1) and reducing, gives

the acceleration along the radius vector

$$= -h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) \tag{2}$$

From these two formulæ the law of acceleration along the radius vector may be deduced when the curve is given, and the curve may be deduced when the law of acceleration along the radius vector is given. Examples of these processes will be given in Chap. (2), Part III.

164. When the Angular Velocity is Constant.— Let the angular velocity be constant $= \omega$ suppose. Then

$$\frac{d\theta}{dt} = \omega;$$

therefore from (5) of Art. 161

the acceleration along the radius vector

$$= \frac{d^2r}{dt^2} - r\omega^2. \tag{1}$$

The acceleration perpendicular to the radius vector

$$= 2\omega\frac{dr}{dt}; \tag{2}$$

and both of these are independent of θ.

The following example is an illustration of these formulæ:

A particle describes a path with a constant angular velocity, and without acceleration along the radius vector; find (1) the equation of the path, and (2) the acceleration perpendicular to the radius vector.

(1) From (1) we have, from the conditions of the question,

$$\frac{d^2r}{dt^2} - \omega^2 r = 0.$$

Integrating we have

$$\frac{dr^2}{dt^2} = \omega^2 (r^2 - a^2),$$

if $r = a$ when $\frac{dr}{dt} = 0$.

Therefore $$\frac{dr}{(r^2 - a^2)^{\frac{1}{2}}} = \omega dt;$$

and $$\log\left[\frac{r + (r^2 - a^2)^{\frac{1}{2}}}{a}\right] = \omega t,$$

if $r = a$ when $t = 0$,

$$\therefore \quad r = \frac{a}{2}(e^{\omega t} + e^{-\omega t}). \tag{3}$$

Also, as $\frac{d\theta}{dt} = \omega$, therefore $\theta = \omega t$, if $\theta = 0$ when $t = 0$. Substituting this value of ωt, we have,

$$r = \frac{a}{2}(e^{\theta} + e^{-\theta}); \tag{4}$$

which is the path described by the particle.

(2) Let Q be the required acceleration perpendicular to the radius vector, then from (2) we have

$$Q = 2\omega \frac{dr}{dt}$$

$$= a\omega^2 (e^{\omega t} - e^{-\omega t}), \text{ from (3)}$$

$$= a\omega^2 (e^\theta - e^{-\theta})$$

$$= 2\omega^2 (r^2 - a^2)^{\frac{1}{2}}; \qquad (5)$$

which is the acceleration perpendicular to the radius vector.

The preceding discussion of Kinematics is sufficient for this work. There are various other problems which might be studied as Kinematic questions, and inserted here; but we prefer to treat them from a Kinetic point of view.

For the investigation of the kinematics of a particle describing a curvilinear path in space, see Price's Anal. Mech's, Vol. I, p. 430, also Tait and Steele's Dynamics of a Particle, p. 12.

EXAMPLES.

1. A particle describes the hyperbola, $xy = k^2$; find (1) the acceleration parallel to the axis of x if the velocity parallel to the axis of y is a constant, β, and (2) find the acceleration parallel to the axis of y if the velocity parallel to the axis of x is a constant, α.

Ans. (1) $\frac{2\beta^2}{k^4} x^3$; (2) $\frac{2\alpha^2}{k^4} y^3$.

2. A particle describes the parabola, $y^2 = 4ax$; find the acceleration parallel to the axis of y if the velocity parallel to the axis of x is a constant, α. *Ans.* $-\frac{4a^2\alpha^2}{y^3}$.

3. A particle describes the logarithmic curve, $y = a^x$; find (1) the x-component of the acceleration if the y-component of the velocity is a constant, β, and (2) find the y-component of the acceleration if the x-component of the velocity is a constant, α.

Ans. (1) $-\frac{\beta^2}{a^{2x}\log a}$; (2) $\alpha^2 (\log a)^2 y$.

4. A particle describes the cycloid, the starting point being the origin; find (1) the x-component of the acceleration if the y-component of the velocity is β, and (2) find the y-component of the acceleration if the x-component of the velocity is α.

Ans. (1) $\dfrac{\beta^2 ay}{(2ay - y^2)^{\frac{3}{2}}}$; (2) $-\dfrac{a\alpha^2}{y^2}$.

5. A particle describes a catenary, $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$; find (1) the x-component of the acceleration if the y-component of the velocity is β, and (2) find the y-component of the acceleration if the x-component of the velocity is α.

Ans. (1) $-\dfrac{\beta^2 ay}{(y^2 - a^2)^{\frac{3}{2}}}$; (2) $\dfrac{\alpha^2}{a^2}y$.

6. Determine how long a particle takes in moving from the point of projection to the further end of the latus rectum.

Ans. $\dfrac{v}{g}(\sin\alpha + \cos\alpha)$.

7. A gun was fired at an elevation of 50°; the ball struck the ground at the distance of 2449 ft.; find (1) the velocity with which it left the gun and (2) the time of flight. $(g = 32\frac{1}{6})$.

Ans. (1) 282.8 ft. per sec.; (2) 13.47 secs.

8. A ball fired with velocity u at an inclination α to the horizon, just clears a vertical wall which subtends an angle, β, at the point of projection; determine the instant at which the ball just clears the wall.

Ans. $\dfrac{u\sin\alpha - \frac{1}{2}gt}{u\cos\alpha} = \tan\beta$.

9. In the preceding example determine the horizontal distance between the foot of the wall and the point where the ball strikes the ground.

Ans. $\dfrac{2u^2}{g}\cos^2\alpha\tan\beta$.

10. At the distance of a quarter of a mile from the bottom of a cliff, which is 120 ft. high, a shot is to be fired which shall just clear the cliff, and pass over it horizontally; find the angle, α, and velocity of projection, v.

Ans. $\alpha = 10^\circ\ 18'$; $v = 490$ ft. per sec.

11. When the angle of elevation is 40° the range is 2449 ft.; find the range when the elevation is $29\frac{1}{2}^\circ$.

Ans. 2131.5 ft.

12. A body is projected horizontally with a velocity of 4 ft. per sec.; find the latus rectum of the parabola described, ($g = 32$). *Ans.* 1 foot.

13. A body projected from the top of a tower at an angle of 45° above the horizontal direction, fell in 5 secs. at a distance from the bottom of the tower equal to its altitude; find the altitude in feet, ($g = 32$). *Ans.* 200 feet.

14. A ball is fired up a hill whose inclination is 15°; the inclination of the piece is 45°, and the velocity of projection is 500 ft. per sec.; find the time of flight before it strikes the hill, and the distance of the place where it falls from the point of projection.*

Ans. $T = 16.17$ secs.; $R = 1.121$ miles.

15. On a descending plane whose inclination is 12°, a ball fired from the top hits the plane at a distance of two miles and a half, the elevation of the piece is 42°; find the velocity of projection. *Ans.* $v = 579.74$ ft. per sec.

16. A body is projected at an inclination α to the horizon; determine when the motion is perpendicular to a plane which is inclined at an angle β to the horizon.

Ans. $\dfrac{u \sin \alpha - gt}{u \cos \alpha} = \pm \cot \beta.$

---

* The range on the inclined plane.

17. Calculate the maximum range, and time of flight, on a descending plane, the angle of depression of which is 15°, the velocity of projection being 1000 ft. per sec.

Ans. Max. range = 7.98 miles; T = 51.34 sec.

18. With what velocity does the ball strike the plane in the last example? *Ans.* V = 1303 feet.

19. If a ship is moving horizontally with a velocity $= 3g$, and a body is let fall from the top of the mast, find its velocity, V, and direction, θ, after 4 secs.

Ans. $V = 5g$; $\theta = \tan^{-1}\frac{4}{3}$.

20. A body is projected horizontally from the top of a tower, with the velocity gained in falling down a space equal to the height of the tower; at what distance from the base of the tower will it strike the ground?

Ans. R = twice the height of the tower.

21. Find the velocity and time of flight of a body projected from one extremity of the base of an equilateral triangle, and in the direction of the side adjacent to that extremity, to pass through the other extremity of the base.

Ans. $v = \sqrt{\dfrac{2ag}{\sqrt{3}}}$; $T = \sqrt{\dfrac{2a\sqrt{3}}{g}}$.

22. Given the velocity of sound, V; find the horizontal range, when a ball, at a given angle of elevation, α, is so projected towards a person that the ball and sound of the discharge reach him at the same instant.

Ans. $\dfrac{2V^2}{g}\tan\alpha$.

23. A body is projected horizontally with a velocity of $4g$ from a point whose height above the ground is $16g$; find the direction of motion, θ, (1) when it has fallen half-way to the ground, and (2) when half the whole time of falling has elapsed.

Ans. (1) $\theta = 45°$; (2) $\theta = \tan^{-1}\dfrac{1}{\sqrt{2}}$.

24. Particles are projected with a given velocity, v, in all lines in a vertical plane from the point O; find the locus of them at a given time, t.

Ans. $x^2 + (y + \frac{1}{2}gt^2)^2 = v^2t^2$, which is the equation of a circle whose radius is vt and whose centre is on the axis of y at a distance $\frac{1}{2}gt^2$ below the origin.

25. How much powder will throw a 13-inch shell* 4000 ft. on an inclined plane whose angle of elevation is 10° 40′; the elevation of the mortar being 35°.

Ans. Charge = 4.67 lbs.

26. A projectile is discharged in a horizontal direction, with a velocity of 450 ft. per sec., from the summit of a conical hill, the vertical angle of which is 120°; at what distance down the hillside will the projectile fall, and what will be the time of flight?

Ans. Distance = 2812.5 yards; Time = 16.23 secs.

27. A gun is placed at a distance of 500 ft. from the base of a cliff which is 200 ft. high; on the edge of the cliff there is built the wall of a castle 60 ft. high; find the elevation, α, of the gun, and the velocity of discharge, v, in order that the ball may graze the top of the castle wall, and fall 120 ft. inside of it.

Ans. $\alpha = 53° 19'$; $v = 165$ ft. per sec.

28. A piece of ordnance burst when 50 yards from a wall 14 ft. high, and a fragment of it, originally in contact with the ground, after grazing the wall, fell 6 ft. beyond it on the opposite side; find how high it rose in the air.

Ans. 94 ft.

---

* **The weight of a 13-inch shell is 196 lbs.**

PART III.

KINETICS (MOTION AND FORCE).

CHAPTER I.

LAWS OF MOTION—MOTION UNDER THE ACTION OF A VARIABLE FORCE—MOTION IN A RESISTING MEDIUM.

165. Definitions.—*Kinetics is that branch of Dynamics which treats of the motion of bodies under the action of forces.*

In Part I, *forces* were considered with reference to the *pressures* which they produced upon bodies at rest (Art. 15), *i. e.*, bodies under the action of two or more forces in equilibrium (Art. 26). In Part II we considered the purely geometric properties of the *motion* of a point or particle without any reference to the causes producing it, or the properties of the thing moved. We are now to consider motion with reference to the causes which produce it, and the things in which it is produced.

The student must here review Chapter I, Part I, and obtain clear conceptions of *Momentum*, *Acceleration of Momentum*, and the *Kinetic measure of Force* (Arts. 12, 13, 19, and 20), as this is necessary to a full understanding of the fundamental laws of motion, on the truth of which all our succeeding investigations are founded.

166. Newton's Laws of Motion.—The fundamental principles in accordance with which motion takes place are embodied in three statements, generally known as *Newton's Laws of Motion.* These laws must be considered as resting on convictions drawn from observation and experiment, and *not* on intuitive perception.* The laws are the following:

LAW I.—*Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it is compelled by force to change that state.*

LAW II.—*Change of motion is proportional to the force applied, and takes place in the direction of the straight line in which the force acts.*

LAW III.—*To every action there is always an equal and contrary reaction; or, the mutual actions of any two bodies are always equal and oppositely directed.*

167. Remarks on Law I.—Law I supplies us with a definition of force. It indicates that force is that which tends to change a body's state of rest or of uniform motion in a straight line; for if a body does not continue in its state of rest or of uniform motion in a straight line it must be under the action of force.

A body has no power to change its own state as to rest or motion; when it is at rest, it has no power of putting itself in motion; when in motion it has no power of increasing or diminishing its velocity. Matter is *inert* (Art. 3). If it is at rest, it will remain at rest; if it is moving with a given velocity along a rectilinear path, it will continue to move with that velocity along that path. It is alike natural to matter to be at rest or in motion. Whenever, therefore, a body's state is changed either from rest to motion, or from motion to rest, or when its velocity is increased or diminished, that change is due to some external cause. This cause is *called force* (Art. 14); and the word *force* is used in Kinetics in this meaning only.

---

* Thomson and Tait's Nat. Phil., p. 241.

168. Remarks on Law II.—Law II asserts that if any force generates motion, a double force will generate double motion, and so on, whether applied simultaneously or successively, instantaneously or gradually. And this motion, if the body was moving beforehand, is either added to the previous motion if directly conspiring with it, or is subtracted if directly opposed; or is geometrically compounded with it according to the principles already explained (Art. 29), if the line of previous motion and the direction of the force are inclined to each other at an angle. The term *motion* here means *quantity of motion*, and the phrase *change of motion* here means *rate of change of quantity of motion* (Art. 13). If the force be finite it will require a finite time to produce a sensible change of motion, and the change of momentum produced by it will depend upon the time during which it acts. The change of motion must then be understood to be the change of momentum produced per unit of time, or the *rate* of change of momentum, or acceleration of momentum, which agrees with the principles already explained (Arts. 13 and 20). In this law nothing is said about the actual motion of the body before it was acted on by the force; it is only the *change of motion* that concerns us. The same force will produce precisely the same change of motion in a body; whether the body be at rest, or in motion with any velocity whatever.

Since, when several forces act at once on a particle either at rest or in motion, the second law of motion is true for *every one* of these forces, it follows that each must have the same effect, in so far as the change of motion produced by it is concerned, as if *it* were the only force in action. Hence the assertion of the second law may be put in the following form:

When any number of forces act simultaneously on a body, whether at rest or in motion in any direction, each force produces in the body the same change of motion as if it alone had acted on the body at rest.

It follows from this view of the law that all problems which involve forces acting simultaneously may be treated as if the forces acted *successively*.

The operations of this law have already been considered in Kinematics (Art. 149); but motion there was understood to mean *velocity* only, since the mass of the body was not considered. This law includes, therefore, the law of the composition of velocities already referred to (Art. 29). Another consequence of the law is the following: Since forces are measured by the changes of motion they produce, and their directions assigned by the directions in which these changes are produced, and since the changes of motion of one and the same body are in the directions of, and proportional to, the changes of velocity, therefore a single force, measured by the resultant change of velocity, and in its direction, will be the equivalent of any number of simultaneously acting forces.

Hence,

The resultant of any number of concurring forces is to be found by the same geometric process as the resultant of any number of simultaneous velocities, and conversely.

From this follows at once the *Polygon of Velocities* and the *Parallelopiped of Velocities* from the Polygon and Parallelopiped of Forces, as was described in Art. 142.

This law also gives us the means of measuring *force*, and also of measuring the *mass* of a body: for the actions of different forces upon the same body for equal times, evidently produce changes of velocity which are *proportional* to the *forces*. Also, if equal forces act on different bodies for equal times, the changes of velocity produced must be *inversely* as the *masses* of the bodies. Again, if different bodies, each acted on by a force, acquire in the same time the same changes of velocity, the forces must be proportional to the masses of the bodies. This means of measuring force is practically the same as that already deduced by abstract reasoning (Arts. 19 and 20).

It appears from this law, that every theorem of Kinematics connected with acceleration has its counterpart in Kinetics. Thus, the measure of acceleration or velocity increment, (Art. 9), which was discussed in Chap. I (Arts. 8 and 9), and in Kinematics (Art. 135), and which is denoted by f or its equal $\frac{d^2s}{dt^2}$, is also the effect and the measure of force; therefore all the results of the equation

$$f = \frac{d^2s}{dt^2}, \qquad (1)$$

its various forms, and the remarks which have been made on it, are applicable to it when f is the accelerating force. Thus, (Art. 162), we see that the force, under which a particle describes any curve, may be resolved into two components, one in the tangent to the curve, the other *towards* the centre of curvature; their magnitudes being the acceleration of momentum, and the product of the momentum into the angular velocity about the centre of curvature, respectively. In the case of uniform motion, the first of these vanishes, or the whole force is perpendicular to the direction of motion. When there is no force perpendicular to the direction of motion, there is no curvature, or the path is a straight line.

Hence if we suppose the particle of mass m to be at the point (x, y, z), and resolve the forces acting on it into the three rectangular components, X, Y, Z, we have

$$m\frac{d^2x}{dt^2} = \mathrm{X} : m\frac{d^2y}{dt^2} = \mathrm{Y}\,; m\frac{d^2z}{dt^2} = \mathrm{Z}. \qquad (2)$$

In several of the chapters these equations will be simplified by assuming unity as the mass of the moving particle. When this cannot be done, it is sometimes convenient to assume X, Y, Z, as the component forces on *the unit mass*, and (2) becomes

$$m\frac{d^2x}{dt^2} = m\mathrm{X}, \text{ etc.}$$

from which m may of course be omitted. It will be observed that an equation such as

$$\frac{d^2x}{dt^2} = \mathrm{X}$$

may be interpreted either as Kinetical or Kinematical; if the former, the unit of mass must be understood as a factor on the left-hand side, in which case X is the x-component, for the unit of mass, of the whole force exerted on the moving body.

The first two laws, have, therefore, furnished us with a *definition* and a *measure* of force; and they also show how to compound, and therefore how to resolve, forces; and also how to investigate the conditions of equilibrium or motion of a single particle subjected to given forces.

169. Remarks on Law III.—According to Law III, if one body presses or draws another, it is pressed or drawn by this other with an equal force in the opposite direction (Art. 16). A horse towing a boat on a canal, is pulled backwards by a force equal to that which he impresses on the towing-rope forwards. If one body strikes another body and changes the motion of the other body, its own motion will be changed in an equal quantity and in the opposite direction; for at each instant during the impact the bodies exert on each other equal and opposite pressures, and the momentum that one body loses is equal to that which the other gains.

The earth attracts a falling pebble with a certain force, while the pebble attracts the earth with an equal force. The result is that while the pebble moves towards the earth on account of its attraction, the earth also moves towards the pebble under the influence of the attraction of the latter; but the mass of the earth being enormously greater than that of the pebble while the forces on the two arising from their mutual attractions are equal, the motion produced thereby in the earth is almost incomparably less than that produced in the pebble, and is consequently insensible.

It follows that the sum of the quantities of motion parallel to any fixed direction of the particles of any system influencing one another in any possible way, remains unchanged by their mutual action. Therefore if the centre of gravity of any system of mutually influencing particles is in motion, it continues moving uniformly in a straight line, unless in so far as the direction or velocity of its motion is changed by forces between the particles and some *other matter not belonging to the system*; also the centre of gravity of any system of particles moves just as all the matter of the system, if concentrated in a point, would move under the influence of forces equal and parallel to the forces really acting on its different parts. (For further remarks on these laws see Tait and Steele's Dynamics of a Particle, Thomson and Tait's Nat. Phil., Pratt's Mechanics, etc.)'

170. Two Laws of Motion in the French Treatises.—Newton's Laws of motion are not adopted in the principal French treatises; but we find in them *two* principles only as borrowed from experience, viz.:

First.—The *Law of Inertia,* that a body, not acted upon by any force, would go on for ever with a uniform velocity. This coincides with Newton's First Law.

Second.—That the *velocity* communicated is *proportional to the force.* The *second* and *third* Laws of Motion are thus reduced to this second principle by the French writers, especially Poisson and Laplace.*

171. Motion of a Particle under the Action of an Attractive Force.—*A particle moves under a force of attraction which is in its line of motion, and varies directly as the distance of the particle from the centre of force; it is required, to determine the motion.*

The point whence the influence of a force emanates is called *the centre of force;* and the force is called an *attractive* or a *repulsive* force according as it *attracts* or *repels.*

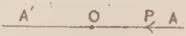

Fig. 80

Let O be the centre of force, P the position of the particle at any time, t, v its velocity at that time, and let $OP = x$, and $OA = a$, where A is the position of the particle when $t = 0$; let $\mu =$ the *absolute force;* that is, the force of attraction on a unit of mass at a unit's distance from O, which is supposed to be known, and is sometimes called the *strength* of the attraction. At present we shall suppose the mass of the particle to be unity, as it simplifies the equations. Then μx is the magnitude of the force at the distance x on the particle of unit mass, or it is the acceleration at P; and the equation of motion is

---

* Parkinson's Mechanics, p. 187. See paper by Dr. Whewell on the principles of Dynamics, particularly as stated by French writers, in the Edinburgh Journal of Science, Vol. VIII.

$$\frac{d^2x}{dt^2} = -\mu x \tag{1}$$

the negative sign being taken because the tendency of the force is to diminish x;

$$\therefore \quad \frac{2dx\,d^2x}{dt^2} = -2\mu x\,dx.$$

Integrating, we get

$$\frac{dx^2}{dt^2} = \mu\,(a^2 - x^2), \tag{2}$$

if the particle be at rest when $x = a$ and $t = 0$,

$$\therefore \quad \frac{-dx}{\sqrt{a^2 - x^2}} = \mu^{\frac{1}{2}}dt,$$

the negative sign being taken, because x decreases as t increases. Integrating again between the limits corresponding to $t = t$ and $t = 0$,

$$\cos^{-1}\frac{x}{a} = \mu^{\frac{1}{2}}t,$$

$$\therefore \quad t = \frac{1}{\mu^{\frac{1}{2}}}\cos^{-1}\frac{x}{a}. \tag{3}$$

From (2) it appears that the velocity of the particle is zero when $x = a$ and $-a$; and is a maximum, viz.: $a\mu^{\frac{1}{2}}$, when $x = 0$. Hence the particle moves from rest at A; its velocity increases until it reaches O where it becomes a maximum, and where the force is zero; the particle passes through that point, and its velocity decreases, and at A′, at a distance $= -a$, becomes zero. From this point it will return, under the action of the force, to its original position, and continually oscillate over the space $2a$, of which O is the middle point.

From (3) we find when $x = a$, $t = 0$ and when $x = 0$, $t = \frac{\pi}{2\mu^{\frac{1}{2}}}$; so that the time of passing from A to O $= \frac{\pi}{2\mu^{\frac{1}{2}}}$, and the time from O to A′ is the same, so that the time of oscillation from A to A′ is $\frac{\pi}{\mu^{\frac{1}{2}}}$. This result is remarkable, as it shows that the time of oscillation is independent of the velocity and distance of projection, and depends solely on the strength of the attraction, and is greater as that is less.

This problem includes the motion of a particle within a homogeneous sphere of ordinary matter in a straight shaft through the centre. For the attraction of such a sphere on a particle within its bounding surface varies directly as the distance from the centre of the sphere (Art. 133a). If the earth were such a homogeneous sphere, and if AOA′ (Fig. 80) represented a shaft running straight through its centre from surface to surface, then, if a particle were free at one end, A, it would move to the centre of the earth, O, where its velocity would be a maximum, and thence on to the opposite side of the earth, A′, where it would come to rest; then it would return through the centre, O, to the side, A, from where it started; and its motion would continue to be oscillatory, and thus it would move backwards and forwards from one side of the earth's surface to the other, and the time of the oscillation would be independent of the earth's radius; that is, at whatever point within the earth's surface the particle be placed it would reach the centre in the same time.

COR.—*To find this time.* Since μ is the attraction at a unit of distance and g the attraction at the distance R, we have $\mu = \frac{g}{R}$, which in $t = \frac{\pi}{2\mu^{\frac{1}{2}}}$ gives

$$t = \frac{\pi}{2}\sqrt{\frac{R}{g}},$$

for the time it would take a body to move from any point within the earth's surface to the centre.

If we put $g = 32\frac{1}{6}$ feet and $R = 3963$ miles we get

$$t = 21 \text{ m. } 6 \text{ s. about},$$

which would be the time occupied in passing to the earth's centre, however near to it the body might be placed, or however far, so long as it is within the surface.

172. Motion of a Particle under the Action of a Variable Repulsive Force.—*Let the force be one of repulsion and vary as the distance,* then the equation of motion is

$$\frac{d^2x}{dt^2} = \mu x.$$

Let us suppose the particle to be projected from the centre of force with the velocity v_0; then we have

$$\frac{dx^2}{dt^2} = \mu x^2 + v_0^2; \qquad (1)$$

$$\therefore \quad x = \frac{v_0}{2\mu^{\frac{1}{2}}}\left(e^{\mu^{\frac{1}{2}}t} - e^{-\mu^{\frac{1}{2}}t}\right).$$

As t increases x also increases, and the particle recedes further and further from the centre of force; and the velocity also increases, and ultimately equals ∞ when $x = t = \infty$. Thus in this case the motion is not oscillatory.

173. Motion of a Particle under the Action of an Attractive Force which is in the line of motion, and which varies Inversely as the Square of the Distance from the Centre of Force.

Let O (Fig. 80) be the centre of force, P the position of the particle at the time t; and A the position at rest when $t = 0$, so that the particle starts from A and moves towards O. Let $OP = x$, $OA = a$, and $\mu =$ the absolute force as before or the acceleration at unit distance from O. Then the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}.$$

Multiplying by $2dx$ and integrating, we get

$$\frac{dx^2}{dt^2} = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right), \tag{1}$$

which gives the velocity of the particle at any distance, x, from the origin.

From (1) we have

$$\frac{dx}{dt} = -\sqrt{\frac{2\mu}{a}}\,\frac{\sqrt{ax - x^2}}{x},$$

the negative sign being taken because in the motion towards O, x diminishes as t increases. This gives

$$\sqrt{\frac{2\mu}{a}}\,dt = \frac{-\,x\,dx}{\sqrt{ax - x^2}}$$

$$= \left[\frac{1}{2}\,\frac{a - 2x}{\sqrt{ax - x^2}} - \frac{a}{2}\,\frac{1}{\sqrt{ax - x^2}}\right]dx.$$

Integrating and taking the limits corresponding to $t = t$ and $t = 0$, we have

$$t = \sqrt{\frac{a}{2\mu}}\left[\sqrt{ax - x^2} - \frac{a}{2}\operatorname{vers}^{-1}\frac{2x}{a} + \frac{\pi a}{2}\right] \qquad (2)$$

which gives the value of t.

When the particle arrives at O, $x = 0$, therefore the time of falling to the centre O from A is

$$t = \frac{\pi}{\sqrt{\mu}}\left(\frac{a}{2}\right)^{\frac{3}{2}}.$$

From (1) we see that the velocity $= 0$ when $x = a$; and $= \infty$ when $x = 0$; hence the velocity increases as the particle approaches the centre of force, and ultimately, when it arrives at the centre, becomes infinite. And although at any point very near to O there is a very great attraction tending towards O, at the point O itself there is no attraction at all; therefore the particle, approaching the centre with an indefinitely great velocity, must pass through it. Also, everything being the same at equal distances on either side of the centre, we see that the motion must be retarded as rapidly as it was accelerated, and therefore the particle will proceed to a point A′ at a distance on the other side of O equal to that from which it started; and the motion will continue oscillatory.

174. Velocity acquired in Falling through a Great Height above the Earth.—The preceding case of motion includes that of a body falling from a great height above the earth's surface towards its centre, the distance through which it falls being so great that the variations of the earth's attraction due to the distance must be taken into account. For a sphere attracts an external particle with a force which varies inversely as the square of the distance of the particle from the centre of the sphere (Art. 133a); therefore if R is the earth's radius, g the kinetic measure of gravity on a unit of mass at the earth's surface (Arts. 20, 23), and x the distance of a body from the centre of the earth at the time t, then the equation of motion is

$$\frac{d^2x}{dt^2} = -g\frac{R^2}{x^2},$$

which is the same as the equation in Art. 173 by writing μ for gR^2; therefore the results of the last Art. will apply to this case. Substituting gR^2 for μ in (1) of Art. 173 we have

$$v^2 = 2gR^2\left(\frac{a-x}{ax}\right). \tag{1}$$

When the body reaches the earth's surface, $x = R$ and (1) becomes

$$v^2 = 2gR\left(\frac{a-R}{a}\right). \tag{2}$$

If a is infinite (2) becomes

$$v = \sqrt{2gR};$$

so that the velocity can never be so great as this, however far the body may fall; and hence if it were possible to project a body vertically upwards with this velocity it would go on to infinity and never stop, supposing, of course, that there is no resisting medium nor other disturbing force.

If in (2) we put $g = 32\frac{1}{6}$ feet and $R = 3963$ miles we get

$$v = [2\cdot 32\tfrac{1}{6}\cdot 3963\cdot 5280]^{\frac{1}{2}} \text{ feet} = 6\cdot 95 \text{ miles};$$

so that the greatest possible velocity which a body can acquire in falling to the earth is less than 7 miles per second, and if a body were projected upwards with that velocity, and were to meet with no resistance except gravity, it would never return to the earth.

Cor.—To find the velocity which a body would acquire in falling to the earth's surface from a height h above the surface, we have from (1) by putting $x = R$ and $a = h + R$,

$$v^2 = 2gR^2\left(\frac{1}{R} - \frac{1}{R+h}\right) = \frac{2gRh}{R+h}.$$

If h be small compared with R, this may be written

$$v^2 = 2gh,$$

which agrees with (6) of Art. 140.

The laws of force, enumerated in Arts. 171, 173, are the only laws that are known to exist in the universe (Pratt's Mechs., p. 212).

175. Motion in a Resisting Medium.—In the preceding discussion no account is taken of the atmospheric resistance. We shall now consider the motion of a body near the surface of the earth, taking into account the resistance of the air, which we may assume varies as the square of the velocity.

A particle under the action of gravity, as a constant force, moves in the air supposed to be a resisting medium of uniform density, of which the resistance varies as the square of the velocity required to determine the motion.

Suppose the particle to descend towards the earth from rest. Take the origin at the starting point, let the line of its motion be the axis of x; and let x be the distance of the particle from the origin at the time t, and for convenience let gk^2 be the resistance of the air on the particle for a unit of velocity; gk^2 is called the *coefficient of resistance.* Then the resistance of the air at the distance x from the origin is $gk^2\left(\frac{dx}{dt}\right)^2$, which acts upwards, and the force of gravity is g acting downwards, the mass being a unit. Hence the equation of motion is

$$\frac{d^2x}{dt^2} = g - gk^2\left(\frac{dx}{dt}\right)^2, \qquad (1)$$

$$\therefore \quad g\,dt = \frac{d\,\dfrac{dx}{dt}}{1 - k^2\left(\dfrac{dx}{dt}\right)^2}.$$

Integrating, remembering that when $t = 0$, $v = 0$, we get

$$gt = \frac{1}{2k}\log\frac{1 + k\,\dfrac{dx}{dt}}{1 - k\,\dfrac{dx}{dt}}, \text{ (Calculus, p. 259, Ex. 5).}$$

Passing to exponentials we have

$$\frac{dx}{dt} = \frac{1}{k}\,\frac{e^{kgt} - e^{-kgt}}{e^{kgt} + e^{-kgt}}, \qquad (2)$$

which gives the velocity in terms of the time. To find it in terms of the space, we have from (1)

$$\frac{k^2 d\left(\dfrac{dx}{dt}\right)^2}{1 - k^2\left(\dfrac{dx}{dt}\right)^2} = 2gk^2dx;$$

$$\therefore \quad \log\left[1 - k^2\left(\frac{dx}{dt}\right)^2\right] = -\,2gk^2x, \qquad (3)$$

observing the proper limits;

$$\therefore \quad \frac{dx^2}{dt^2} = \frac{1}{k^2}(1 - e^{-2gk^2x}), \tag{4}$$

which gives the velocity in terms of the distance.

Also, integrating (2) taking the same limits as before, we get

$$gk^2x = \log(e^{kgt} + e^{-kgt}) - \log 2;$$

$$\therefore \quad 2e^{gk^2x} = e^{kgt} + e^{-kgt}, \tag{5}$$

which gives the relation between the distance and the time of falling through it.

As the time increases the term e^{-kgt} diminishes and from (5) the space increases, becoming infinite when the time is infinite; but from (2), as the time increases the velocity becomes more nearly uniform, and when $t = \infty$, the velocity $= \frac{1}{k}$; and although this state is never reached, yet it is that to which the motion approaches.

176. Motion of a Particle Ascending in the Air against the Action of Gravity.—Let us suppose the particle to be projected upwards, that is, in a direction contrary to that of the action of gravity, with a given velocity, v, it is required to determine the motion.

Let us suppose the particle to be of the same form and size as before, and the same coefficient of resistance. Then, taking x positive upwards, both gravity and the resistance of the air tend to diminish the velocity as t increases; so that the equation of motion is

$$\frac{d^2x}{dt^2} = -g - gk^2\left(\frac{dx}{dt}\right)^2; \tag{1}$$

$$\therefore \quad \frac{dk\dfrac{dx}{dt}}{1 + k^2\left(\dfrac{dx}{dt}\right)^2} = -kg\,dt;$$

$$\therefore \quad \tan^{-1} k\frac{dx}{dt} = \tan^{-1}(kv) - gkt;$$

(Calculus, p. 244, Ex. 3), since the initial velocity is v.

Taking the tangent of both members and solving for $\frac{dx}{dt}$, we get

$$\frac{dx}{dt} = \frac{1}{k}\cdot\frac{vk - \tan kgt}{1 + vk\tan kgt}; \tag{2}$$

which gives the velocity in terms of the time. To find it in terms of the distance, we have from (1)

$$\frac{dk^2\left(\dfrac{dx}{dt}\right)^2}{1 + k^2\left(\dfrac{dx}{dt}\right)^2} = -2gk^2\,dx;$$

$$\therefore \quad \log\frac{1 + k^2\left(\dfrac{dx}{dt}\right)^2}{1 + k^2v^2} = -2gk^2x; \tag{3}$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = v^2e^{-2gk^2x} - \frac{1}{k^2}\left(1 - e^{-2gk^2x}\right), \tag{4}$$

which gives the velocity in terms of the distance.

Also, integrating (2) after substituting sine and cosine for tangent, and taking the same limits as before, we get

$$gk^2x = \log(vk\sin kgt + \cos kgt); \tag{5}$$

which gives the space described by the particle in terms of the time.

COR. 1.—To find the greatest height to which the particle will ascend put the velocity, $\frac{dx}{dt} = 0$, in (3) and get

$$x = \frac{1}{2gk^2} \log (1 + k^2v^2), \tag{6}$$

which is the distance of the highest point.

Putting $\frac{dx}{dt} = 0$ in (2) we get

$$t = \frac{1}{kg} \tan^{-1} vk, \tag{7}$$

which is the time required for the particle to reach the highest point. Having reached the greatest height, the particle will begin to fall, and the circumstances of the fall will be given by the equations of Art. 175.

COR. 2.—Since k is the same in this and Art. 175, we may compare the velocity of projection, v, with that which the particle would acquire in descending to the point whence it was projected. Denote by v_0 the velocity of the particle when it reaches the point of starting. From (3) of Art. 175 we have

$$x = \frac{1}{2gk^2} \log \frac{1}{1 - k^2{v_0}^2},$$

and placing this value of x equal to that given in (6), we get,

$$\frac{1}{1 - k^2{v_0}^2} = 1 + k^2v^2;$$

$$\therefore \quad v_0 = \frac{v}{(1 + k^2v^2)^{\frac{1}{2}}};$$

which is less than v; hence the velocity acquired in the descent is less than that lost in the ascent, as might have been inferred.

Cor. 3.—Substituting (6) in (5) of Art. 175, we get for the time of the descent,

$$t = \frac{1}{kg} \log \left(\sqrt{1 + k^2v^2} + kv\right),$$

which is different from the time of the ascent as given in (7). (See Price's Anal. Mech's, Vol. I, p. 406; Venturoli's Mech's, p. 82; Tait and Steele's Dynamics of a Particle, p. 237.)

177. Motion of a Projectile in a Resisting Medium.—The theory of the motion of projectiles in vacuo, which was examined under the head of Kinematics, affords results which differ greatly from those obtained by direct experiment in the atmosphere. When projectiles move with but small velocity, the discrepancy between the parabolic theory, and what is found to occur in practice, is small; but with increasing velocities, as those with which balls and shells traverse their paths, the air's resistance increases in a higher ratio than the velocity, so that the discrepancy becomes very great.

The most important application of the theory of projectiles, is that of Gunnery, in which the motion takes place in the air. If it were allowable to neglect the resistance of the air the investigations in Part II would explain the theory of gunnery; but when the velocity is considerable, the atmospheric resistance changes the nature of the trajectory so much as to render the conclusions drawn from the theory of projectiles in vacuo almost entirely inapplicable in practice.

The problem of gunnery may be stated as follows: Given a projectile of known weight and dimensions, starting with a known velocity at a known angle of elevation in a calm atmosphere of approximately known density; to find its range, time of flight, velocity, direction, and position, at any moment; or, in other words, to construct its trajectory. This problem is not yet, however, susceptible of rigorous treatment; mathematics has hitherto proved unable to furnish complete formulæ satisfying the conditions. The resistance of the air to slow movements, say of 10 feet per second, seems to vary with the first power of the velocity. Above this the ratio increases, and as in the case of the wind, is usually reckoned to vary as the square of the velocity; beyond this it increases still further, till at 1200 feet per second the resistance is found to vary as the cube of the velocity. The ratio of increase after this point is passed is supposed to diminish again; but thoroughly satisfactory data for its determination do not exist.

From experiments* made to determine the motion of cannon-balls, it appears that when the initial velocity is considerable, the resistance of the air is more than 20 times as great as the weight of the ball, and the horizontal range is often a small fraction of that which the theory of projectiles in vacuo gives, so that the form of the trajectory is very different from that of a parabolic path. Such experiments have been made with great care, and show how little the parabolic theory is to be depended upon in determining the motion of military projectiles.

178. Motion of a Projectile in the Atmosphere Supposing its Resistance to vary as the Square of the Velocity.—*A particle under the action of gravity is projected from a given point in a given direction with a given velocity, and moves in the atmosphere whose resistance is assumed to vary as the square of the velocity; to determine the motion.*

---

* See Encyclopædia Britannica, Art. Gunnery; also Robin's Gunnery, and Hutton's Tracts.

Take the given point as origin, the axis of x horizontal, the axis of y vertical and positive upwards, so that the direction of projection may be in the plane of xy. Let v be the velocity of projection, g the acceleration of gravity, α the angle between the axis of x and the line of projection, and let the resistance of the air on the particle be k for a unit of velocity; then the resistance, at any time, t, in the line of motion, is $k\left(\frac{ds}{dt}\right)^2$; and the x- and y-components of this resistance are, respectively,

$$k\frac{ds}{dt}\cdot\frac{dx}{dt}, \quad \text{and} \quad k\frac{ds}{dt}\cdot\frac{dy}{dt}.$$

Then the equations of motion are, resolving horizontally and vertically,

$$\frac{d^2x}{dt^2} = -k\frac{ds}{dt}\ \frac{dx}{dt}, \tag{1}$$

$$\frac{d^2y}{dt^2} = -g - k\frac{ds}{dt}\ \frac{dy}{dt}. \tag{2}$$

From (1) we have

$$\frac{d\left(\frac{dx}{dt}\right)}{\frac{dx}{dt}} = -kds; \quad \therefore \quad \log\frac{\frac{dx}{dt}}{v\cos\alpha} = -ks;$$

since when $t = 0$, $\frac{dx}{dt} = v\cos\alpha$;

$$\therefore \quad \frac{dx}{dt} = v\cos\alpha\, e^{-ks}. \tag{3}$$

Multiplying (1) and (2) by dy and dx, respectively, and subtracting the former from the latter we have

$$\frac{d^2y\,dx - d^2x\,dy}{dt^2} = -gdx. \tag{4}$$

Substituting in (4) for dt^2 its value from (3) we get

$$\frac{d^2y\,dx - d^2x\,dy}{dx^2} = d\frac{dy}{dx} = -\frac{g}{v^2\cos^2\alpha}e^{2ks}dx. \qquad (5)$$

Substituting in the second member of (5) for dx its value $ds \div \sqrt{1 + \frac{dy^2}{dx^2}}$, we get

$$d\frac{dy}{dx}\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = -\frac{g}{v^2\cos^2\alpha}e^{2ks}ds. \qquad (6)$$

Put $\frac{dy}{dx} = p$, and (6) becomes

$$(1 + p^2)^{\frac{1}{2}}\,dp = -\frac{g}{v^2\cos^2\alpha}e^{2ks}\,ds.$$

Integrating, and remembering that when $s = 0, p = \tan\alpha$, we get

$$p\,(1 + p^2)^{\frac{1}{2}} + \log\left[p + (1 + p^2)^{\frac{1}{2}}\right]$$
$$= c - \frac{g}{kv^2\cos^2\alpha}e^{2ks}. \qquad (7)$$

where c is the constant of integration whose value

$$= \tan\alpha\sec\alpha + \log(\tan\alpha + \sec\alpha) + \frac{g}{kv^2\cos^2\alpha}. \qquad (8)$$

From (5) we have

$$-\frac{g}{v^2\cos^2\alpha}e^{2ks} = \frac{d}{dx}\left(\frac{dy}{dx}\right);$$

which in (7) gives

$$p\,(1 + p^2)^{\frac{1}{2}} + \log\left[p + (1 + p^2)^{\frac{1}{2}}\right] - c = \frac{1}{k}\frac{dp}{dx},$$

$$\therefore\ \frac{dp}{p\,(1 + p^2)^{\frac{1}{2}} + \log\left[p + (1 + p^2)^{\frac{1}{2}}\right] - c} = k\,dx, \qquad (9)$$

and $$\frac{pdp}{p\,(1+p^2)^{\frac{1}{2}}+\log\,[p+(1+p^2)^{\frac{1}{2}}]-c}=kdy. \quad (10)$$

From (4) we have

$$dx\cdot dp=-gdt^2.$$

Substituting this value of dx in (9) and solving for dt we get

$$\frac{-dp}{\{c-p\,(1+p^2)^{\frac{1}{2}}-\log\,[p+(1+p^2)^{\frac{1}{2}}]\}^{\frac{1}{2}}}=(kg)^{\frac{1}{2}}\,dt. \quad (11)$$

the negative sign of dp being taken because p is a decreasing function of t.

Replacing the value of $p=\frac{dy}{dx}$, (9), (10), and (11) become

$$dx=\frac{1}{k}\,\frac{d\frac{dy}{dx}}{\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}+\log\left[\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right]-c}, \quad (A)$$

$$dy=\frac{1}{k}\,\frac{\frac{dy}{dx}\,d\frac{dy}{dx}}{\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}+\log\left[\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right]-c}, \quad (B)$$

$$dt=\frac{1}{(kg)^{\frac{1}{2}}}\,\frac{-d\frac{dy}{dx}}{\left\{c-\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}-\log\left[\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}}, \quad (C)$$

from which equations, were it possible to integrate them, x, y, and t might be found in terms of $\frac{dy}{dx}$; and if $\frac{dy}{dx}$ were eliminated from the two integrals, of (A) and (B), the resulting equation in terms of x and y would be that of the required trajectory. But these equations cannot be integrated in finite terms; only approximate solutions of them can be made; and by means of these the path of the projectile may be constructed approximately. (See Venturoli's Mechs., p. 92.)

Squaring (A) and (B), and dividing their sum by the square of (C) we get

$$\frac{ds^2}{dt^2} = \frac{g}{k} \frac{1 + \frac{dy^2}{dx^2}}{c - \frac{dy}{dx}\left(1 + \frac{dy^2}{dt^2}\right)^{\frac{1}{2}} - \log\left[\frac{dy}{dx} + \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right]} \qquad \text{(D)}$$

which gives the velocity in terms of $\frac{dy}{dx}$.

179. Motion of a Projectile in the Atmosphere under a small Angle of Elevation.—The case frequently occurs in practice where the angle of projection is very small, and where the projectile rises but a very little above the horizontal line. In this case the equation of the part of the trajectory that lies above the horizontal line may easily be found; for, the angle of projection being very small, $\frac{dy}{dx}$ will be very small, and therefore, throughout the path on the upper side of the axis of x, powers of $\frac{dy}{dx}$ higher than the first may be neglected. In this case then

$$ds = dx; \qquad \therefore \; s = x;$$

which in (5) of Art. 178, becomes

$$d\frac{dy}{dx} = -\frac{g}{v^2 \cos^2 \alpha} e^{2kx} dx;$$

Integrating, we get

$$\frac{dy}{dx} - \tan\alpha = -\frac{g}{2kv^2\cos^2\alpha}(e^{2kx} - 1);$$

since when $x = 0$, $\frac{dy}{dx} = \tan\alpha$.

Integrating again we get

$$y = x\tan\alpha + \frac{gx}{2kv^2\cos^2\alpha} - \frac{g}{4k^2v^2\cos^2\alpha}(e^{2kx} - 1). \quad (1)$$

Expanding e^{2kx} in a series, (1) becomes

$$y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha} - \frac{gkx^3}{3v^2\cos^2\alpha} - \dots \quad (2)$$

the first two terms of which represent the trajectory in vacuo. [See (3) of Art. 151.]

From (3) of Art. 178, we have

$$dt = \frac{e^{kx}}{v\cos\alpha}\,dx\cdot$$

$$\therefore\ t = \frac{e^{kx} - 1}{kv\cos\alpha} \quad (3)$$

which gives the time of flight in terms of the abscissa.

The most complete and valuable series of experiments on the motion of projectiles in the atmosphere that has yet been made, is that of Prof. F. Bashforth at Woolwich.

EXAMPLES.

1. Find how far a force equal to the weight of n lbs., would move a weight of m lbs. in t seconds; and find the velocity acquired.

Here $P = n$, and $W = m$; therefore from (1) of Art. 25 we have

$$n = \frac{m}{g}f; \quad \therefore\ f = \frac{ng}{m},$$

which in (4) and (5) respectively of (Art. 9), gives $v = \frac{ngt}{m}$; and $s = \frac{1}{2}\frac{ng}{m}t^2$.

2. A body weighing n lbs. is moved by a constant force which generates in the body in one second a velocity of a feet per second; find the force in pounds. *Ans.* $\frac{na}{g}$ lbs.

3. Find in what time a force of 4 lbs. would move a weight of 9 lbs. through 49 ft. along a smooth horizontal plane; and find the velocity acquired.

Ans. $t = \frac{21}{\sqrt{2g}}$; $v = \frac{4}{9}gt$.

4. Find the number of inches through which a force of one ounce, constantly exerted, will move a mass weighing one lb. in half a second. *Ans.* $3g\left(\frac{1}{2}\right)^5$.

5. Two weights, P and Q, are connected by a string which passes over a smooth peg or pulley; required to determine the motion.

Since the peg or pulley is perfectly smooth the tension of the string is the same throughout; hence the force which causes the motion is the difference between the weights, P and Q, the weight of the string being neglected. The moving force therefore is $P - Q$; but the weight of the mass moved is $P + Q$. Hence substituting in (1) of Art. 25, we get

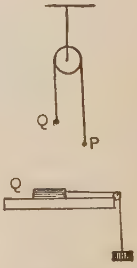

Fig. 80*a*.

$$P - Q = \frac{P + Q}{g}f;$$

$$\therefore\ f = \frac{P - Q}{P + Q} g. \qquad (1)$$

which is the acceleration.

Substituting this in (4) and (5) of Art. 9, we have

$$v = \frac{P - Q}{P + Q} gt, \qquad (2)$$

$$s = \tfrac{1}{2} \frac{P - Q}{P + Q} gt^2 \qquad (3)$$

which gives the velocity and space at the time t, the initial velocity v_0 being 0.

6. A body whose weight is Q, rests on a smooth horizontal table and is drawn along by a weight P attached to it by a string passing over a pulley at the edge of the table; find the motion of the bodies.

Since the weight Q is entirely supported by the resistance of the table, the moving force is the weight P, hanging vertically downwards, and the weight of the mass moved is $P + Q$; therefore from (1) we have

$$f = \frac{P}{P + Q} \cdot g \qquad (1)$$

and this in (4) and (5) of Art. 9 gives the velocity and space.

7. Required the tension, T, of the string in the preceding example.

Here the tension is evidently that force which, acting along the string on the body whose weight is Q, produces in it the acceleration, $\frac{P}{P + Q} g$, and therefore is measured by the mass of Q into its acceleration. Hence

$$T = \frac{Q}{g} \times \frac{P}{P+Q} g = \frac{PQ}{P+Q}.$$

8. Find the tension, T, of the string in Ex. 5.

Here the tension equals the weight Q, plus the force which, acting along the string on Q, produces in it the acceleration

$$\frac{P-Q}{P+Q} g,$$

$$\therefore \quad T = Q + \frac{Q}{g} \cdot \frac{P-Q}{P+Q} g,$$

$$= \frac{2PQ}{P+Q},$$

or it equals P minus the accelerating force which, of course, gives the same result.

9. Two weights of 9 lbs. and 7 lbs. hang over a pulley, as in Ex. 5; motion continues for 5 secs., when the string breaks; find the height to which the lighter weight will rise after the breakage.

Substituting in (2) of Ex. 5 we have

$$v = \tfrac{2}{16} 32 \cdot 5 = 20;$$

therefore each weight has a velocity of 20 feet, when the string breaks. Hence from (6) of Art. 9, we have (calling g 32 ft.)

$$s = \tfrac{400}{64} = 6\tfrac{1}{4};$$

that is, the lighter weight will rise $6\frac{1}{4}$ feet before it begins to descend.

10. A steam engine is moving on a horizontal plane at the rate of 30 miles an hour when the steam is turned off; supposing the resistance of friction to be $\frac{1}{400}$ of the weight, find how long and how far the engine will run before it stops.

Let W be the weight of the engine; then the resistance of friction is $\frac{W}{400}$, and this is directly opposed to motion,

$$\therefore \quad \frac{W}{400} = \frac{W}{g} f; \qquad \therefore \quad f = \frac{g}{400}.$$

The velocity, v, is 30 miles an hour $= \frac{30 \times 1760 \times 3}{60 \times 60} = 44$ feet per second. Substituting these values of f and v in the equation $v = ft$, we get

$$44 = \tfrac{32}{400} t;$$

$$\therefore \quad t = 550 \text{ secs.},$$

which is the time it will take to bring the engine to rest if the velocity be retarded $\frac{32}{400}$ feet per second.

Also $v^2 = 2fs$, therefore

$$s = \tfrac{44 \times 44 \times 400}{64} = 12100 \text{ feet.}$$

11. A man whose weight is W, stands on the platform of an elevator, as it descends a vertical shaft with a uniform acceleration of $\frac{1}{2}g$; find the pressure of the man upon the platform.

Let P be the pressure of the man on the platform when it is moving with an acceleration of $\frac{1}{2}g$; then the moving force is $W - P$; and the weight moved is W; therefore

$$W - P = \frac{W}{g} \tfrac{1}{2} g; \qquad \therefore \quad P = \tfrac{1}{2} W.$$

12. A plane supporting a weight of 12 ozs. is descending with a uniform acceleration of 10 ft. per second; find the pressure that the weight exerts on the plane.

Ans. $8\frac{1}{4}$ ozs.

13. A weight of 24 lbs. hanging over the edge of a smooth table drags a weight of 12 lbs. along the table; find (1) the acceleration, and (2) the tension of the string.

Ans. (1) $21\frac{1}{3}$ ft. per sec.; (2) 8 lbs.

14. A weight of 8 lbs. rests on a platform; find its pressure on the platform (1) if the latter is descending with an acceleration of $\frac{1}{8}g$, and (2) if it is ascending with the same acceleration.

Ans. (1) 7 lbs.; (2) 9 lbs.

15. Two weights of 80 and 70 lbs. hang over a smooth pulley as in Ex. 5; find the space through which they will move from rest in 3 secs. *Ans.* $9\frac{3}{5}$ ft.

16. Two weights of 15 and 17 ounces respectively hang over a smooth pulley as in Ex. 5; find the space described and the velocity acquired in five seconds from rest.

Ans. $s = 25$, $v = 10$.

17. Two weights of 5 lbs. and 4 lbs. together pull one of 7 lbs. over a smooth fixed pulley, by means of a connecting string; and after descending through a given space the 4 lbs. weight is detached and taken away without interrupting the motion; find through what space the remaining 5 lbs. weight will descend.

Ans. Through $\frac{3}{4}$ of the given space.

18. Two weights are attached to the extremities of a string which is hung over a smooth pulley, and the weights are observed to move through 6.4 feet in one second; the motion is then stopped, and a weight of 5 lbs. is added to the smaller weight, which then descends through the same space as it ascended before in the same time; determine the original weights. *Ans.* $\frac{9}{8}$ lbs.; $\frac{21}{8}$ lbs.

19. Find what weight must be added to the smaller weight in Ex. 5, so that the acceleration of the system may

have the same numerical value as before, but may be in the opposite direction. *Ans.* $\frac{P^2 - Q^2}{Q}$.

20. A body is projected up a rough inclined plane with the velocity which would be acquired in falling freely through 12 feet, and just reaches the top of the plane; the inclination of the plane to the horizon is 60°, and the coefficient of friction is equal to tan 30°; find the height of the plane. *Ans.* 9 feet.

21. A body is projected up a rough inclined plane with the velocity $2g$; the inclination of the plane to the horizon is 30°, and the coefficient of friction is equal to tan 15°; find the distance along the plane which the body will describe. *Ans.* $g(\sqrt{3} + 1)$.

22. A body is projected up a rough inclined plane; the inclination of the plane to the horizon is α, and the coefficient of friction is tan ϵ; if m be the time of ascending, and n the time of descending, show that

$$\left(\frac{m}{n}\right)^2 = \frac{\sin(\alpha - \epsilon)}{\sin(\alpha + \epsilon)}.$$

23. A weight P is drawn up a smooth plane inclined at an angle of 30° to the horizon, by means of a weight Q which descends vertically, the weights being connected by a string passing over a small pulley at the top of the plane; if the acceleration be one-fourth of that of a body falling freely, find the ratio of Q to P. *Ans.* $Q = P$.

24. Two weights P and Q are connected by a string, and Q hanging over the top of a smooth plane inclined at 30° to the horizon, can draw P up the length of the plane in just half the time that P would take to draw up Q; show that Q is half as heavy again as P.

25. A particle moves in a straight line under the action of an attraction varying inversely as the $(\frac{3}{2})$th power of the distance; show that the velocity acquired by falling from an infinite distance to a distance a from the centre is equal to the velocity which would be acquired in moving from rest at a distance a to a distance $\frac{a}{4}$.

CHAPTER II.

CENTRAL FORCES.*

180. Definitions.—*A central force* is one which acts directly towards or from a fixed point, and is called an *attractive* or a *repulsive* force according as its action on any particle is *attraction* or *repulsion*. The fixed point is called the *Centre*. The intensity of the force on any particle is some function of its distance from the centre. Since the case of *attraction* is the most important application of the subject, we shall take that as our standard case; but it will be seen that a simple change of sign will adapt our general formulæ to repulsion. If the centre be itself in motion, we may treat it as fixed, in which case the term "actual motion" of any particle means its motion "relative" to the centre, taken as fixed.

The line from the centre to the particle, is called a *Radius Vector*. The path of the particle under the action of an attraction or repulsion directed to the centre is called its *Orbit*.† All the forces of nature with which we are acquainted, are central forces; for this reason, and because the motion of bodies under the action of central forces is a branch of the general theory of Astronomy, we shall devote this chapter to the consideration of their action.

181. A Particle under the Action of a Central Attraction; Required the Polar Equation of the Path.—The motion will clearly take place in the plane passing through the centre, and the line along which the particle is initially projected, as there is nothing to withdraw the particle from it. Let the centre of attraction, O, be the origin, and OX, OY, any two lines through O at right angles to each other, be the axes of co-ordinates. Let (x, y) be the position of the particle M at the time t, and (r, θ) its position referred to polar co-ordinates, OX being the initial line. Then, calling P the central attractive force, we have for the components parallel to the axes of x and y, respectively, $-P\frac{x}{r}$, $-P\frac{y}{r}$, the forces being negative, since they tend to diminish the co-ordinates. Therefore the equations of motion are

---

* This chapter contains the first principles of Mathematical Astronomy. It may, however, be omitted by the student of Engineering.

† Called Central Orbits.

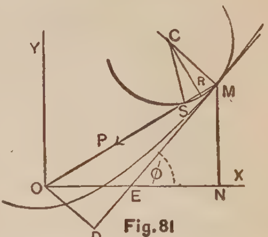

Fig. 81

$$\frac{d^2x}{dt^2} = -P\frac{x}{r}, \quad \frac{d^2y}{dt^2} = -P\frac{y}{r}. \tag{1}$$

Multiplying the former by y, and the latter by x, and subtracting, we have

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0. \tag{2}$$

Integrating we have

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h; \tag{3}$$

where h is an undetermined constant.

Since $x = r\cos\theta$, and $y = r\sin\theta$, we have

$$dx = \cos\theta\,dr - r\sin\theta\,d\theta,$$

$$dy = \sin\theta\,dr + r\cos\theta\,d\theta, \tag{4}$$

which in (3) gives

$$r^2 \frac{d\theta}{dt} = h. \tag{5}$$

Again, multiplying the first and second of (1) by $2dx$ and $2dy$ respectively, and adding, we get

$$\frac{2dx\, d^2x + 2dy\, d^2y}{dt^2} = -\frac{2P\,(x\,dx + y\,dy)}{r};$$

$$\therefore \quad d\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right) = -\,2Pdr. \tag{6}$$

Substituting in (6) the values of dx^2 and dy^2 from (4), we have

$$d\left[\left(\frac{dr^2}{d\theta^2} + r^2\right)\frac{d\theta^2}{dt^2}\right] = -\,2Pdr;$$

$$\therefore \quad d\left(\frac{1}{r^4}\frac{dr^2}{d\theta^2} + \frac{1}{r^2}\right) = -\frac{2P}{h^2}\,dr, \text{ by (5).} \tag{7}$$

Put $r = \frac{1}{u}$; and $\therefore\ dr = -\frac{du}{u^2}$; and (7) becomes

$$d\left(\frac{du^2}{d\theta^2} + u^2\right) = \frac{2P}{h^2u^2}\,du;$$

performing the differentiation of the first member, and dividing by $2du$, and transposing, we get

$$\frac{d^2u}{d\theta^2} + u - \frac{P}{h^2u^2} = 0. \tag{8}$$

which is the differential equation of the orbit described; and as, in any particular instance, the force P will be given in terms of r, and therefore in terms of u, the integral of this equation will be the polar equation of the required path.

Solving (8) for P we have

$$P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right); \tag{9}$$

which is the same result that was found by a different process in Art. 163 for the acceleration along the radius vector.

Cor. 1.—The general integrals of (1) will contain four arbitrary constants. One, h, that was introduced in (5), and two more will be introduced by the integration of (8). If the value of r in terms of θ, deduced from the integral of (8), be substituted in (5), and that equation be then integrated, the fourth constant will be introduced, and the path of the particle and its position at any time will be obtained. The four constants must be determined from the initial circumstances of motion; viz., the initial position of the particle, depending on *two* independent co-ordinates, its initial velocity, and its direction of projection.

Cor. 2.—By means of (9) we may ascertain the law of the force which must act upon a particle to cause it to describe a given curve. To effect this we must determine the relation between u and θ from the polar equation of the orbit referred to the required centre as pole; we must then differentiate u twice with respect to θ, and substitute the result in the expression for P, eliminating θ, if it occurs, by means of the relation between u and θ. In this way we shall obtain P in terms of u alone, and therefore of r alone.

Cor. 3.—When we know the relation between r and θ from (9), we may by (5) determine the time of describing a given portion of the orbit; or, conversely, find the position of the particle in its orbit at any time.*

---

* See Tait and Steele's Dynamics of a Particle, p. 119; also Pratt's Mech's, p. 222.

Cor. 4.—If p is the perpendicular from the origin to the tangent we have from Calculus, p. 176,

$$x\,dy - y\,dx = p\,ds;$$

which in (3) gives

$$\frac{ds}{dt} = \frac{h}{p}; \tag{10}$$

and this in (6) gives

$$d\frac{h^2}{p^2} = -2Pdr.$$

Differentiating, and solving for P, we have

$$P = \frac{h^2}{p^3}\frac{dp}{dr}, \tag{11}$$

which is the equation of the orbit between the radius vector and the perpendicular on the tangent at any point.

182. The Sectorial Area Swept over by the Radius Vector of the Particle in any time is Proportional to the Time.—Let A denote this area; then we have from Calculus, p. 364,

$$\begin{aligned} A &= \tfrac{1}{2}\int r^2\,d\theta \\ &= \tfrac{1}{2}\int h\,dt, \text{ by (5) of Art. 181,} \\ &= \tfrac{1}{2}ht, \end{aligned}$$

if A and t be both measured from the commencement of the motion. *Therefore the areas swept over by the radius vector in different times are proportional to the times, and equal areas will be described in equal times.*

Cor.—If $t = 1$, we have $A = \frac{1}{2}h$. Hence h = twice the sectorial area described in one unit of time.

183. The Velocity of the Particle at any Point of its Orbit.—We have for the velocity,

$$v = \frac{ds}{dt}$$

$$= \frac{h}{p} \text{ by (10) of Art. 181.} \tag{1}$$

Hence, *the velocity of the particle at each point of its path is inversely proportional to the perpendicular from the centre on the tangent at that point.*

COR. 1.—We have, by Calculus, p. 180,

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\frac{dr^2}{d\theta^2}$$

$$= u^2 + \frac{du^2}{d\theta^2}, \text{ since } r = \frac{1}{u} \text{ (Art. 181),}$$

which in (1) gives

$$v^2 = \frac{h^2}{p^2} = h^2\left(u^2 + \frac{du^2}{d\theta^2}\right), \tag{2}$$

another important expression for the velocity.

COR. 2.—From (6) of Art. 181, we have

$$d\left(\frac{ds^2}{dt^2}\right) = d\,(v^2) = -\,2Pdr. \tag{3}$$

Let V be the velocity at the point of projection, at which let $r = R$, and since P is some function of r, let $P = f(r)$, then integrating (3) we get

$$\frac{ds^2}{dt^2} = -\,2\int_R^r f(r)\,dr;$$

$$\therefore \quad v^2 - V^2 = 2\,[f_1(R) - f_1(r)], \tag{4}$$

which is another expression for the velocity; and since this is a function only of the corresponding distances, R and r, it follows that *the velocity at any point of the orbit is* *independent of the path described, and depends solely on the magnitude of the attraction, the distance of the point from the centre, and the velocity and distance of projection.*

From (4) it appears that the velocity is the same at all points of the same orbit which are equally distant from the centre; if $r = R$, the velocity $= V$; and thus if the orbit is a re-entering curve, the particle always, in its successive revolutions, passes through the same point with the same velocity.

If the velocity vanishes at a distance a from the centre (4) becomes

$$v^2 = 2\,[f_1(a) - f_1(r)] \tag{5}$$

and a is called the radius of the circle of zero velocity.

Cor. 3.—From (3) we have

$$d\,(v^2) = -\,2Pdr;$$

$$\therefore\ \ vdv = -\,Pdr. \tag{6}$$

Taking the logarithm of (1) we have

$$\log v = \log h - \log p.$$

Differentiating we get

$$\frac{dv}{v} = -\,\frac{dp}{p}. \tag{7}$$

Dividing (6) by (7), we get

$$v^2 = Pp\,\frac{dr}{dp} = 2P\cdot\frac{p}{2}\,\frac{dr}{dp}$$

$= 2P \times \frac{1}{4}$ chord of curvature* through the centre; (8)

---

* To prove that $\frac{p}{2}\frac{dr}{dp}$ is one-fourth the chord of curvature.

Let MD (Fig. 81), be the tangent to the orbit, and C the centre of curvature; let OD $= p$, CM $= \rho$, the radius of curvature; and the angle MEN $= \phi$. Then MS, the and, comparing this with (6) of Art. 140, it appears that the particle at any point has the same velocity which it would have if it moved from rest at that point towards the centre of force, under the action of the force continuing constant, through one-fourth of the chord of the circle of curvature.

Hence, *the velocity of a particle at any point of a central orbit is the same as that which would be acquired by a particle moving freely from rest through one-fourth of the chord of curvature at that point, through the centre, under the action of a constant force whose magnitude is equal to that of the central attraction at the point.*

Cor. 4.—If the orbit is a circle having the centre of force

---

part of the radius vector OM, which is intercepted by the circle of curvature is called the *chord of curvature*. Its value is determined as follows:

We have (Fig. 81)

$$\phi = \theta + \text{OMD}$$

$$= \theta + \sin^{-1}\frac{p}{r};$$

$$\therefore \quad d\phi = d\theta + \frac{r\,dp - p\,dr}{r\sqrt{r^2 - p^2}}. \qquad (1)$$

From Calculus, p. 180, (10), we have

$$d\theta = \frac{p\,dr}{r\sqrt{r^2 - p^2}}, \qquad (2)$$

and

$$ds = \frac{r^2\,d\theta}{p} = \frac{r\,dr}{\sqrt{r^2 - p^2}}. \qquad (3)$$

Substituting (2) in (1) we get

$$d\phi = \frac{dp}{\sqrt{r^2 - p^2}}. \qquad (4)$$

But Calculus, p. 221, we have

$$\rho = \frac{ds}{d\phi} = r\frac{dr}{dp}, \text{ by (3) and (4).} \qquad (5)$$

Now MS (Fig. 81) = 2MC sin OMD,

$$= 2\rho\frac{p}{r} = 2p\frac{dr}{dp}, \text{ by (5)}$$

= the chord of curvature; therefore

$$\frac{p}{2}\frac{dr}{dp} = \text{one-fourth the chord of curvature.}$$

in the centre, and R, V, P, are respectively the radius, velocity and central force, we have

$$V^2 = PR.$$

Cor. 5.—From (5) of Art. 181, we have

$$\frac{d\theta}{dt} = \frac{h}{r^2}. \tag{9}$$

The first member, being the actual velocity of a point on the radius vector at the unit's distance from the centre, is the angular velocity of the particle (Art. 160). *Hence the angular velocity of a particle varies inversely as the square of the radius vector.*

Sch.—A point in a central orbit at which the radius vector is a maximum or minimum is called an *Apse;* the radius vector at an apse is called an *Apsidal Distance;* and the angle between two consecutive apsidal distances is called an *Apsidal Angle* of the orbit. The analytical conditions for an apse are, of course, that $\frac{du}{d\theta} = 0$, and that the first derivative which does not vanish should be of an even order. The first condition ensures that the radius vector at an apse is perpendicular to the tangent.

184. The Orbit when the Attraction Varies Inversely as the Square of the Distance.—*A particle is projected from a given point in a given direction with a given velocity, and moves under the action of a central attraction varying inversely as the square of the distance; to determine the orbit.*

Let the centre of force be the origin; $V =$ the velocity of projection; $R =$ the distance of the point of projection from the origin; $\beta =$ the angle between R and the line of projection; and let $\mu =$ the absolute force and $t = 0$ when the particle is projected. Then since the velocity $= \frac{h}{p}$ (Art. 183), and at the point of projection $p = R \sin \beta$, we have

$$V = \frac{h}{R \sin \beta}; \; h = VR \sin \beta. \qquad (1)$$

As the force varies inversely as the square of the distance, we have

$$P = \frac{\mu}{r^2} = \mu u^2, \left(\text{since } r = \frac{1}{u}\right). \qquad (2)$$

which in (9) of Art. 181 gives

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}. \qquad (3)$$

Multiplying by $2du$ and integrating, we get

$$\frac{du^2}{d\theta^2} + u^2 = 2\frac{\mu}{h^2}u + c;$$

when $t = 0$, $u = \frac{1}{r} = \frac{1}{R}$, and $\frac{du^2}{d\theta^2} + u^2 = \frac{V^2}{h^2}$, (Art. 183, Cor. 1); therefore

$$c = \frac{V^2}{h^2} - \frac{2\mu}{h^2 R} = \frac{V^2 R - 2\mu}{h^2 R}.$$

Substituting this value for c we get

$$\frac{du^2}{d\theta^2} + u^2 = \frac{V^2 R - 2\mu}{h^2 R} + \frac{2\mu u}{h^2}. \qquad (4)$$

Therefore (Art. 183, Cor. 1) we have

$$(\text{velocity})^2 = V^2 + 2\mu\left(\frac{1}{r} - \frac{1}{R}\right) \qquad (5)$$

which shows that the velocity is the greatest when r *is the least, and the least when* r *is the greatest.*

Changing the form of (4) we have

$$\frac{du^2}{d\theta^2} = \frac{V^2R - 2\mu}{h^2R} + \frac{\mu^2}{h^4} - \left(\frac{\mu}{h^2} - u\right)^2. \qquad (6)$$

To express this in a simpler form, let

$$\frac{\mu}{h^2} = b, \text{ and } \frac{V^2R - 2\mu}{h^2R} + \frac{\mu^2}{h^4} = c^2; \text{ and (6) becomes}$$

$$\frac{du^2}{d\theta^2} = c^2 - (u - b)^2;$$

$$\therefore \quad \frac{-du}{[c^2 - (u - b)^2]^{\frac{1}{2}}} = d\theta,$$

the negative sign of the radical being taken. Integrating we have,

$$\cos^{-1}\frac{u - b}{c} = \theta - c',$$

where c' is an arbitrary constant;

$$\therefore \quad u = b + c\cos(\theta - c'). \qquad (7)$$

Replacing in (7) the values of b and c, and the value of h, from (1), and dividing both terms of the second member by μ, we have for the equation of the path,

$$u = \frac{1 + \left[\frac{1}{\mu^2}(V^2R - 2\mu)\,RV^2\sin^2\beta + 1\right]^{\frac{1}{2}}\cos(\theta - c')}{\frac{R^2V^2\sin^2\beta}{\mu}} \qquad (8)$$

which is the equation of a conic section, the pole being at the focus, and the angle $(\theta - c')$ being measured from the shorter length of the axis major. For if e is the eccentricity of a conic section, r the focal radius vector, and ϕ the angle between r and that point of a conic section which is nearest the focus, we have,

$$\frac{1}{r} = u = \frac{1 + e \cos \phi}{1 \sim e^2}. \tag{9}$$

Comparing (8) and (9), we see that

$$e^2 = \frac{1}{\mu^2} (V^2 R - 2\mu) R V^2 \sin^2 \beta + 1; \tag{10}$$

$$\phi = \theta - c'. \tag{11}$$

Now the conic section is an ellipse, parabola, or hyperbola, according as e is less than, equal to, or greater than unity; and from (10) e is less than, equal to, or greater than, unity according as $V^2 R - 2\mu$ is negative, zero, or positive; therefore we see that if

$$V^2 < \frac{2\mu}{R}, \; e < 1, \text{ and the orbit is an ellipse,} \tag{12}$$

$$V^2 = \frac{2\mu}{R}, \; e = 1, \text{ and the orbit is a parabola,} \tag{13}$$

$$V^2 > \frac{2\mu}{R}, \; e > 1, \text{ and the orbit is a hyperbola.} \tag{14}$$

Cor. 1.—By (1) of Art. 173, we see that the square of the velocity of a particle falling from infinity to a distance R from the centre of force, for the law of attraction we are considering, is $\frac{2\mu}{R}$. Hence the above conditions may be expressed more concisely by saying that *the orbit, described about this centre of force, will be an ellipse, a* *parabola, or a hyperbola, according as the velocity is less than, equal to, or greater than, the velocity from infinity.*

The *species* of conic section, therefore, does not depend on the position of the line in which the particle is projected, but on the *velocity* of projection in reference to the distance of the point of projection from the centre of force.

Cor. 2.—From (11), we see that $\theta - c'$ is the angle between the focal radius vector, r, and that part of the principal axis which is between the focus and the point of the orbit which is nearest to the focus; *i. e.*, it is the angle PFA (Fig. 82); and therefore if the principal axis is the initial line $c' = 0$.

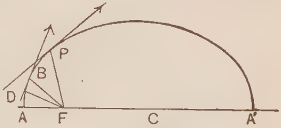

Fig. 82

185. Suppose the Orbit to be an Ellipse.—Here $V^2 < \frac{2\mu}{R}$; so that from (10) we have

$$e^2 = 1 - \frac{1}{\mu^2}(2\mu - V^2R)\,RV^2\sin^2\beta. \qquad (1)$$

Now the equation of an ellipse, where r is the focal radius vector, θ the angle between r and the shorter segment of the major axis, $2a$ the major axis, e the eccentricity, is

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta};$$

$$\therefore\ \ u = \frac{1}{a(1 - e^2)} + \frac{e\cos\theta}{a(1 - e^2)}; \qquad (2)$$

comparing (2) with (8) of Art. 184, we have

$$\frac{1}{a(1 - e^2)} = \frac{\mu}{R^2V^2\sin^2\beta};$$

substituting for $1 - e^2$ its value from (1), and solving for a, we have

$$a = \frac{\mu R}{2\mu - V^2 R}, \tag{3}$$

which shows that the major axis is independent of the direction of projection.

We may explain the several quantities which we have used, by Fig. 82.

B is the point of projection; FB $= R$; DB is the line along which the particle is projected with the velocity V; FBD $= \beta$, the angle of projection; FP $= r$; PFA $= \theta$; FD $= R \sin \beta$; if $\beta = 90°$, the particle is projected from an apse, *i. e.*, from A or A′.

Cor. 1.—To determine the apsidal distances, FA and FA′, we must put $\frac{du}{d\theta} = 0$, (Art. 183, Sch.), and (4) of Art. 184 give us the quadratic equation

$$u^2 - \frac{2\mu}{h^2} u + \frac{2\mu}{h^2 R} - \frac{V^2}{h^2} = 0. \tag{4}$$

the two roots of which are the reciprocals of the two apsidal distances, $a(1 - e)$ and $a(1 + e)$.

Cor. 2.—Since the coefficient of the second term of (4) is the sum of the roots with their signs changed, we have

$$\frac{1}{a(1-e)} + \frac{1}{a(1+e)} = \frac{2\mu}{h^2};$$

$$\therefore \quad a(1 - e^2) = \frac{h^2}{\mu}; \tag{5}$$

which gives the latus rectum of the orbit.

COR. 3.—From Art. 182 we have, calling T the time,

$$T = \frac{2\text{A}}{h},$$

where A is the area swept over by the radius vector in the time T. Therefore for the time of describing an ellipse, we have

$$T = \frac{2 \text{ area of ellipse}}{h}$$

$$= \frac{2\pi a^2 \sqrt{1 - e^2}}{\sqrt{a\mu\,(1 - e^2)}}, \text{ from (5)},$$

$$= 2\pi \sqrt{\frac{a^3}{\mu}},$$

*which is the time occupied by the particle in passing from any point of the ellipse around to the same point again.**

186. Kepler's Laws.—By laborious calculation from an immense series of observations of the planets, and of Mars in particular, Kepler enunciated the following as the laws of the planetary motions about the Sun.

I. The orbits of the planets are ellipses, of which the Sun occupies a focus.

II. The radius vector of each planet describes equal areas in equal times.

III. The squares of the periodic times of the planets are as the cubes of the major axes of their orbits.

187. To Determine the Nature of the Force which Acts upon the Planetary System.—(1) From the second of these laws it follows that the planets are retained in their orbits by an attraction tending to the Sun.

---

* Called *Periodic Time.*

Let (x, y) be the position of a planet at the time t referred to two co-ordinate axes drawn through the Sun in the plane of motion of the planet; X, Y, the component accelerations due to the attraction acting on it, resolved parallel to the axes; then the equations of motions are

$$\frac{d^2x}{dt^2} = X; \quad \frac{d^2y}{dt^2} = Y;$$

$$\therefore \quad x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = xY - yX. \tag{1}$$

But, by Kepler's second law, if A be the area described by the radius vector, $\frac{d\mathrm{A}}{dt}$ is constant,

$$\therefore \quad \frac{d\mathrm{A}}{dt} \quad \text{or} \quad \tfrac{1}{2}\frac{r^2d\theta}{dt}$$

$$= \tfrac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \text{a constant.}$$

Differentiating, we have

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0.$$

$$\therefore \quad xY - yX = 0, \text{ from (1)},$$

$$\therefore \quad \frac{X}{Y} = \frac{x}{y},$$

which shows that the axial components of the acceleration, due to the attraction acting on the planet, are proportional to the co-ordinates of the planet; and therefore, by the parallelogram of forces (Art. 30), the resultant of X and Y passes through the origin.

Hence *the forces acting on the planets all pass through the Sun's centre.*

(2) From the first of these laws it follows that the central attraction varies inversely as the square of the distance.

The polar equation of an ellipse, referred to its focus, is

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta},$$

or
$$u = \frac{1 + e\cos\theta}{a(1 - e^2)}.$$

Hence
$$\frac{d^2u}{d\theta^2} + u = \frac{1}{a(1 - e^2)};$$

and therefore, if P is the attraction to the focus, we have [Art. 181, (9)],

$$P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$= \frac{h^2}{a(1 - e^2)}\,\frac{1}{r^2}.$$

Hence, *if the orbit be an ellipse, described about a centre of attraction at the focus, the law of intensity is that of the inverse square of the distance.*

(3) From the third law it follows that the attraction of the Sun (supposed fixed) which acts on a unit of mass of each of the planets, is the same for each planet at the same distance.

By Art. 185, Cor. 3, we have

$$T^2 = \frac{4\pi^2}{\mu}a^3.$$

But by the third law, $T^2 \propto a^3$, and therefore μ must be constant; *i. e.*, the strength of attraction of the Sun must be the same for all the planets. Hence, not only is the law of force the same for all the planets, but the *absolute* force is the same.

This very brief discussion of central forces is all that we have space for. To pursue these enquiries further would compel us to omit matters that are more especially entitled to a place in this book. The student who wishes to pursue the study further is referred to Tait and Steele's Dynamics of a Particle, or Price's Anal. Mech's, Vol. I, or to any work on Mathematical Astronomy. We shall conclude with the following examples.

EXAMPLES.

1. A particle describes an ellipse under an attraction always directed to the centre; it is required to find the law of the attraction, the velocity at any point of the orbit, and the periodic time.

(1) The polar equation of the ellipse, the pole at the centre, is

$$u^2 = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}; \tag{1}$$

$$\therefore \quad u\frac{du}{d\theta} = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin\theta\cos\theta, \tag{2}$$

and $$u\frac{d^2u}{d\theta^2} + \frac{du^2}{d\theta^2} = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)(\cos^2\theta - \sin^2\theta). \tag{3}$$

But [Art. 181, (9)] we have

$$P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = \frac{h^2}{u}\left(u^4 + u^3\frac{d^2u}{d\theta^2}\right)$$

$$= \frac{h^2}{u}\left\{u^4 + u^2\left[-\frac{du^2}{d\theta^2} + \left(\frac{1}{b^2} - \frac{1}{a^2}\right)(\cos^2\theta - \sin^2\theta)\right]\right\},$$

by (3),

$$= \frac{h^2}{u}\left[u^4 - \left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2 \sin^2\theta\cos^2\theta + u^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\right.$$

$$\left.(\cos^2\theta - \sin^2\theta)\right], \text{ by (2)},$$

$$= \frac{h^2}{u}\left[u^2 + \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cos^2\theta\right]\left[u^2 - \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin^2\theta\right],$$

by factoring,

$$= \frac{h^2}{u}\cdot\frac{1}{b^2}\cdot\frac{1}{a^2}, \text{ by (1)}, = \frac{h^2}{a^2b^2}r, \tag{4}$$

and therefore the attraction varies directly as the distance. If μ = the absolute force we have, by (4),

$$h^2 = \mu\, a^2b^2. \tag{5}$$

(2) If v = the velocity, we have, by Art. 183,

$$v^2 = \frac{h^2}{p^2} = \frac{h^2b'^2}{a^2b^2} \text{ (Anal. Geom., p. 133)}$$

$$= \mu b'^2, \text{ by (5)},$$

where b' is the semi-diameter conjugate to r.

$$\therefore\quad v = b'\sqrt{\mu}.$$

(3) If T = the periodic time, we have, by Art. 182,

$$T = \frac{2\pi ab}{h} = \frac{2\pi}{\sqrt{\mu}}, \text{ by (5)},$$

and hence the periodic time is independent of the magnitude of the ellipse, and depends only on the absolute central attraction. (See Tait and Steele's Dynamics of a

Particle, p. 144, also Price's Anal. Mech's, Vol. I, p. 516.)

2. A particle describes an ellipse under an attraction always directed to one of the foci; it is required to find the law of attraction, the velocity, and the periodic time.

(1) Here we have

$$u = \frac{1 + e\cos\theta}{a(1 - e^2)}; \quad \therefore \quad \frac{du}{d\theta} = -\frac{e\sin\theta}{a(1 - e^2)}; \qquad (1)$$

and

$$\frac{d^2u}{d\theta^2} = \frac{-e\cos\theta}{a(1 - e^2)},$$

which in (9) of Art. 181 gives

$$P = \frac{h^2u^2}{a(1 - e^2)} = \frac{h^2}{a(1 - e^2)} \cdot \frac{1}{r^2}; \qquad (2)$$

hence the attraction varies inversely as the square of the distance. If μ = the absolute force, we have by (2)

$$h^2 = \mu a(1 - e^2). \qquad (3)$$

(2) By Art. 183, Cor. 1, we have

$$\frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2} = \frac{2au - 1}{a^2(1 - e^2)}, \text{ by (1)}; \qquad (4)$$

$$\therefore \quad v^2 = \frac{h^2}{p^2} = \frac{\mu(2au - 1)}{a}, \text{ by (3) and (4)}. \qquad (5)$$

(3) If T = the periodic time we have (Art. 182)

$$T = \frac{2\pi a^2(1 - e^2)^{\frac{1}{2}}}{h}$$

$$= \frac{2\pi a^2(1 - e^2)^{\frac{1}{2}}}{[\mu a(1 - e^2)]^{\frac{1}{2}}} = \frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}}, \qquad (6)$$

and hence the periodic time varies as the square root of the cube of the major axis.

3. Find the attraction by which a particle may describe a circle, and also the velocity, and the periodic time, (1) when the centre of attraction is in the centre of the circle, and (2) when the centre of attraction is in the circumference.

(1) Let $a =$ the radius; then the polar equation, the pole at the centre, is

$$r = a; \quad \therefore \quad u = \frac{1}{a}; \quad \frac{du}{d\theta} = \frac{d^2u}{d\theta^2} = 0;$$

$$\therefore \quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = \frac{h^2}{a^3}. \tag{1}$$

Also $$v^2 = \frac{h^2}{a^2}, \quad \text{and} \quad T = \frac{2\pi a^2}{h}. \tag{2}$$

From (1) and (2) we have

$$P = \frac{v^2}{a},$$

and hence the central attraction is equal to the square of the velocity divided by the radius of the circle.*

(2) The equation, is

$$r = 2a \cos\theta; \quad \therefore \quad 2au = \sec\theta,$$

and $$u + \frac{d^2u}{d\theta^2} = 8a^2u^3;$$

$$\therefore \quad P = 8a^2h^2u^5 = \frac{8a^2h^2}{r^5};$$

and hence the attraction varies inversely as the fifth power of the distance; and if $\mu =$ the absolute force, we have $\mu = 8a^2h^2$;

---

* Called the *Centrifugal Force*. See Art. 198.

$$\therefore \quad h^2 = \frac{\mu}{8a^2}; \quad \text{and} \quad v^2 = \frac{\mu}{2r^4}.$$

If $T =$ the periodic time, we have

$$T = \frac{2^{\frac{5}{2}}\pi a^3}{\mu^{\frac{1}{2}}}. \quad \text{(See Price's Anal. Mech., Vol. I., p. 518.)}$$

4. Find the attraction by which a particle may describe the lemniscate of Bernouilli and also the velocity, and the time of describing one loop, the centre of attraction being in the centre of the lemniscate, and the equation being $r^2 = a^2 \cos 2\theta$.

$$\textit{Ans.}\ P = \frac{3h^2a^4}{r^7}; \ v^2 = \frac{\mu}{3r^6}; \ T = \left(\frac{3}{\mu}\right)^{\frac{1}{2}} a^4.$$

5. Find the attraction by which a particle may describe the cardioid and also the velocity, and the periodic time, the equation being $r = a(1 + \cos\theta)$.

$$\textit{Ans.}\ P = \frac{3ah^2}{r^4}; \ v^2 = \frac{2\mu}{3r^3}; \ T = \left(\frac{3^3a^5}{\mu}\right)^{\frac{1}{2}} \pi.$$

6. Find the attraction by which a particle may describe a parabola, and also the velocity, the centre of attraction being at the focus, and the equation being $r = \dfrac{2a}{1 + \cos\theta}$.

$$\textit{Ans.}\ P = \frac{h^2}{2ar^2}; \ v^2 = \frac{2\mu}{r}. \quad \text{Compare (13) of Art. 184.}$$

7. Find the attraction by which a particle may describe a hyperbola, and the velocity, the centre of attraction being at the focus, and the equation being $r = \dfrac{a(e^2 - 1)}{1 + e\cos\theta}$.

$$\textit{Ans.}\ P = \frac{h^2}{a(1 - e^2)}\frac{1}{r^2}; \ v^2 = \frac{\mu(2au + 1)}{a}.$$

8. If the centre of attraction is at the centre of the hyperbola, find the attraction, and velocity, the equation being $\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} = u^2$.

$$\textit{Ans.}\ P = -\frac{h^2}{a^2b^2}r = -\mu r;\ v^2 = \mu(r^2 - a^2 + b^2).$$

9. Find the attraction to the pole under which a particle will describe (1) the curve whose equation is $r = 2a\cos n\theta$, and (2) the curve whose equation is $r = \frac{2a}{1 - e\cos n\theta}$.

Ans. (1) $P = \frac{8a^2h^2n^2}{r^5} + \frac{(1-n^2)h^2}{r^3}$; (2) $P = \frac{h^2n^2}{2ar^2} + \frac{(1-n^2)h^2}{r^3}$. That is, the attraction in the first curve varies partly as the inverse fifth power, and partly as the inverse cube, of the distance; and in the second it varies partly as the inverse square, and partly as the inverse cube, of the distance.

10. A planet revolved round the sun in an orbit with a major axis four times that of the earth's orbit; determine the periodic time of the planet. *Ans.* 8 years.

11. If a satellite revolved round the earth close to its surface, determine the periodic time of the satellite.

Ans. $\frac{1}{(60)^{\frac{3}{2}}}$ of the moon's period.

12. A body describes an ellipse under the action of a force in a focus: compare the velocity when it is nearest the focus with its velocity when it is furthest from the focus.

Ans. As $1 + e : 1 - e$, where e is the eccentricity.

13. A body describes an ellipse under the action of a force to the focus S; if H be the other focus show that the

velocity at any point P may be resolved into two velocities, respectively at right angles to SP and HP, and each varying as HP.

14. A body describes an ellipse under the action of a force in the centre: if the greatest velocity is three times the least, find the eccentricity of the ellipse. *Ans.* $\frac{2}{3}\sqrt{2}$.

15. A body describes an ellipse under the action of a force in the centre: if the major axis is 20 feet and the greatest velocity 20 feet per second, find the periodic time.

Ans. π seconds.

16. Find the attraction to the pole under which a particle may describe an equiangular spiral. *Ans.* $P \propto \frac{1}{r^3}$.

17. If $P = \frac{\mu}{r^5}(5r^2 - 8c^2)$, and a particle be projected from an apse at a distance c with the velocity from infinity; prove that the equation of the orbit is

$$r = \frac{c}{2}(e^{2\theta} + e^{-2\theta}).$$

18. If $P = 2\mu\left(\frac{1}{r^3} - \frac{a^2}{r^5}\right)$, and the particle be projected from an apse at a distance a with velocity $\frac{\sqrt{\mu}}{a}$, prove that it will be at a distance r after a time

$$\frac{1}{2\sqrt{\mu}}\left(a^2 \log \frac{r + \sqrt{r^2 - a^2}}{a} + r\sqrt{r^2 - a^2}\right).$$

CHAPTER III.

CONSTRAINED MOTION.

188. Definitions.—A particle is *constrained* in its motion when it is compelled to move along a given fixed curve or surface. Thus far the subjects of motion have been particles not constrained by any geometric conditions, but free to move in such paths as are due to the action of the impressed forces. We come now to the case of the motion of a particle which is constrained; that is, in which the motion is subject, not only to given forces, but to undetermined reactions. Such cases occur when the particle is in a small tube, either smooth or rough, the bore of which is supposed to be of the same size as the particle; or when a small ring slides on a curved wire, with or without friction; or when a particle is fastened to a string, or moves on a given surface. If we substitute for the curve or surface a force whose intensity and direction are exactly equal to those of the reaction of the curve, the particle will describe the same path as before, and we may treat the problem as if the particle were free to move under the action of this system of forces, and therefore apply to it the general equations of motion of a free particle.

189. Kinetic Energy or Vis Viva (Living Force), and Work.—*A particle is constrained to move on a given smooth plane curve, under given forces in the plane of the curve, to determine the motion.*

Let APC be the curve along which the particle is compelled to move when acted upon by any given forces. Let Ox and Oy be the rectangular axes in the plane of the

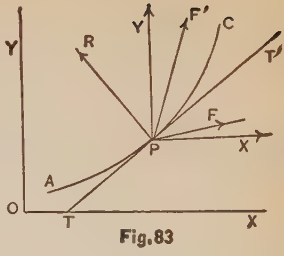

Fig. 83

curve, the axis y positive upwards, and (x, y) the place of the particle, P, at the time t; let X, Y, parallel respectively to the axes of x and y, be the axial components of the forces, the mass of the particle being m; let R be the pressure between the curve and particle, which acts in the normal to the curve, since it is smooth. Then the equations of motion are

$$m\frac{d^2x}{dt^2} = X - R\frac{dy}{ds}; \qquad (1)$$

$$m\frac{d^2y}{dt^2} = Y + R\frac{dx}{ds}. \qquad (2)$$

Multiplying (1) and (2) respectively by dx and dy, and adding, we have

$$m\frac{dx\,d^2x + dy\,d^2y}{dt^2} = Xdx + Ydy.$$

Integrating between the limits t and t_0, and calling v_0 the initial velocity, we have

$$\frac{m}{2}v^2 - \frac{m}{2}v_0{}^2 = \int_{t_0}^{t}(Xdx + Ydy) \qquad (3)$$

The term $\frac{m}{2}v^2$ is called the *vis viva**, or Kinetic Energy of the mass m; that is, vis viva or kinetic energy is a quantity which varies as the product of the mass of the particle and the square of its velocity. There is particular advantage in defining vis viva, or kinetic energy, as *half* the product of the mass and the square of its velocity.* The first member, therefore, of (3) is the vis viva or kinetic energy of m acquired in its motion from (x_0, y_0) to (x, y) under the action of the given forces.

---

* See Thomson and Tait's Nat. Phil., p. 222.

The terms Xdx and Ydy are the products of the axial components of the forces by the axial displacements of the mass in the time dt, and are therefore, the elements of *work* done by the accelerating forces X and Y in the time dt, according to the definition of work given in Art. 101, Rem.; so that the second member of (3) expresses the work done by these forces through the spaces over which they moved the mass in the time between t_0 and t. This equation is called *the equation of kinetic energy and of work;* it shows that the work done by a force exerting action through a given distance, is equal to the increase of kinetic energy which has accrued to the mass in its motion through that distance.

If in the motion, kinetic energy is lost, negative work is done by the force; *i. e.*, the work is stored up as *potential* work in the mass on which the force has acted. Thus, if work is spent on winding up a watch, that work is stored in the coiled spring, and is thus potential and ready to be restored under adapted circumstances. Also, if a weight is raised through a vertical distance, work is spent in raising it, and that work may be recovered by lowering the weight through the same vertical distance.

This theorem, in its most general form, is the modern principle of *conservation of energy;* and is made the fundamental theorem of abstract dynamics as applied to natural philosophy.

In this case we have an instance of *space-integrals,* which, as we have seen, gives us kinetic energy and work; the solution of problems of kinetic energy and work will be explained in Chap. V.

---

* Some writers define vis viva as the whole product of the mass and the square of the velocity. See Routh's Rigid Dynamics, p. 259.

Now if X and Y are functions of the co-ordinates x and y the second member of (3) can be integrated; let it be the differential of some function of x and y, as $\phi(x, y)$. Integrating (3) on this hypothesis, and supposing v and v_0 to be the velocities of the particle at the points (x, y) and (x_0, y_0) corresponding to t and t_0, we have

$$\frac{m}{2}(v^2 - v_0^2) = \phi(x, y) - \phi(x_0, y_0) \tag{4}$$

which shows that the kinetic energy gained by the particle constrained to move, under the forces X, Y, along any path whatever, from the point (x_0, y_0) to the point (x, y), is entirely independent of the path pursued, and depends only upon the co-ordinates of the points left and arrived at; the reaction R does not appear, which is clearly as it should be, since it does no work, because it acts in a line perpendicular to the direction of motion.

190. To Find the Reaction of the Constraining Curve.—For convenience, the mass of the particle may be taken as unity. Multiplying (1) and (2) of Art. 189 by $\frac{dy}{ds}$ and $\frac{dx}{ds}$, subtracting the former from the latter, and solving for R, we have,

$$R = \frac{d^2y\,dx - d^2x\,dy}{dt^2\,ds} + X\frac{dy}{ds} - Y\frac{dx}{ds}$$

$$= \frac{v^2}{\rho} + X\frac{dy}{ds} - Y\frac{dx}{ds}, \text{ by (3) of Art. 162} \tag{1}$$

in which ρ is the radius of curvature at the point P. The last two terms of (1) are the normal components of the impressed forces; and therefore, if the particle were at rest, they would denote the whole pressure on the curve; but the particle being in motion, there is an additional pressure on the curve expressed by $\frac{v^2}{\rho}$.

In the above reasoning we have considered the particle to be on the *concave* side of the curve, and the resultant of X and Y to act towards the *convex* side along some line as PF so as to produce pressure against the curve. If on the contrary, this resultant acts towards the *concave* side, along PF' for example, then, whether the particle be on the concave or convex side, the pressure against the curve will be the difference between $\frac{v^2}{\rho}$ and the normal resultant of X and Y.

191. To Find the Point where the Particle Will Leave the Constraining Curve.—It is evident that at that point, $R = 0$, as there will be no pressure against the curve. Therefore (1) of Art. 190 becomes

$$\frac{v^2}{\rho} = Y\frac{dx}{ds} - X\frac{dy}{ds}$$

$$= F' \cos F'PR$$

if F' be the resultant of X and Y.

$$\therefore \quad v^2 = F'\rho \cos F'PR$$

$$= 2F' \cdot \tfrac{1}{4} \text{ chord of curvature in the direction } PF'.$$

Comparing this with (6) of Art. 140, we see that *the particle will leave the curve at the point where its velocity is such as would be produced by the resultant force then acting on it, if continued constant during its fall from rest through a space equal to ¼ of the chord of curvature parallel to that resultant.* (See Tait and Steele's Dynamics of a Particle, p. 170.)

192. Constrained Motion Under the Action of Gravity.—When gravity is the only force acting on the particle, the formulæ are simplified. Taking the axis of y vertical and positive downwards, the forces become

$$X = 0, \quad \text{and} \quad Y = +g;$$

and for the velocity we have, by (3) of Art. 189,

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = g(y - y_0) \tag{1}$$

where y_0 is the initial space corresponding to the time t_0.

For the pressure on the curve we have, by (1) of Art. 190,

$$R = \frac{v^2}{\rho} + g\frac{dx}{ds}. \tag{2}$$

If the origin be where the motion of the particle begins, the initial velocity and space are zero, and (1) becomes

$$\tfrac{1}{2}v^2 = gy. \tag{3}$$

This shows that the velocity of the particle at any time is entirely independent of the form of the curve on which it moves; and depends solely on the perpendicular distance through which it falls.

193. Motion on a Circular Arc in a Vertical Plane.—Take the vertical diameter as axis of y, and its lower extremity as origin; then the equation of the circle is

$$x^2 = 2ay - y^2;$$

$$\therefore \quad \frac{dx}{a - y} = \frac{dy}{x} = \frac{ds}{a}. \tag{1}$$

Let (k, h) be the point K where the particle starts from rest, and (x, y) the point P where it is at the time t. Then the particle will have fallen through the height $HM = h - y$, and hence from (3) of Art. 192 we have

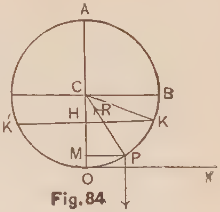

Fig. 84

$$\frac{ds}{dt} = v = \sqrt{2g(h-y)}. \qquad (2)$$

Hence the velocity is a minimum when $y = h$, and a maximum when $y = 0$; and this maximum velocity will carry the particle through O to K' at the distance h above the horizontal line through O.

To find the time occupied by the particle in its descent from K to the lowest point, O, we have from (2)

$$dt = -\frac{ds}{\sqrt{2g(h-y)}}$$

$$= \frac{-a\,dy}{\sqrt{2g(h-y)(2ay-y^2)}} \text{ by (1)} \qquad (3)$$

the negative sign being taken since t is a decreasing function of s.

This expression does not admit of integration; it may be reduced to an *elliptic integral* of the first kind, and tables are given of the approximate values of the integral for given values of y.*

If, however, the radius of the circle is large, and the greatest distance KO, over which the particle moves, is small, we may develope (3) into a series of terms in ascending powers of $\frac{y}{2a}$, and thus find the integral approximately

---

* See Legendre's Traité des Fonctions Elliptiques.

Let T be the time of motion of the particle from K to K', *i. e.*, from $y = h$, through $y = 0$, to $y = h$ again, then (3) becomes

$$T = -\sqrt{\frac{a}{g}}\int_h^0 \frac{dy}{\sqrt{hy - y^2}}\left(1 - \frac{y}{2a}\right)^{-\frac{1}{2}}$$

$$= \sqrt{\frac{a}{g}}\int_0^h \left[1 + \tfrac{1}{2}\frac{y}{2a} + \tfrac{1}{2}\tfrac{3}{4}\left(\frac{y}{2a}\right)^2 + \dots\right]\frac{dy}{\sqrt{hy - y^2}};$$

integrating each term separately we have

$$T = \pi\sqrt{\frac{a}{g}}\left[1 + (\tfrac{1}{2})^2\frac{h}{2a} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{h}{2a}\right)^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(\frac{h}{2a}\right)^3 + \text{etc.}\right] \tag{4}$$

which is the complete expression for the time of moving from the extreme position K on one side of the vertical to the extreme position K' on the other; this is called an oscillation. (See Price's Anal. Mechs., Vol. I., p. 518.)

If the arc is very small, h is very small in comparison with a, and all the terms containing $\frac{h}{2a}$ will be very small, and by neglecting them (4) becomes

$$T = \pi\sqrt{\frac{a}{g}}. \tag{5}$$

194. The Simple Pendulum.—Instead of supposing the particle to move on a *curve*, we may imagine it suspended by a string of invariable length, or a thin rod considered of no weight, and moving in a vertical plane about the point C; for, whether the force acting on the particle be the reaction of the curve or the tension of the string, its *intensity* is the same, while its *direction*, in either case is along the normal to the curve.

When the particle is supposed to be suspended by a thread without weight, it becomes what is termed a *simple pendulum;* and although such an instrument can never be perfectly attained, but exists only in theory, yet approximations may be made to it sufficiently near for practical purposes, and by means of Dynamics we may reduce the calculation of the motion of such a pendulum to that of the simple pendulum.

If l is the length of the rod, the time of an oscillation is approximately given by the formula

$$T = \pi\sqrt{\frac{l}{g}} \qquad (1)$$

when the angle of oscillation is very small, *i. e.*, not exceeding about 4°;* and therefore, for all angles between this and zero, the times of oscillation of the same pendulum will not perceptibly differ; *i. e., in very small arcs the oscillations may be regarded as isochronal,* or as all performed in the same time.

195. Relation of Time, Length, and Force of Gravity.

—From (1) of Art. 194, we have $T \propto \sqrt{l}$ if g is constant; $T \propto \frac{1}{\sqrt{g}}$ if l is constant; $g \propto l$ if T is constant, that is

(1) For the same place *the times of oscillation are as the square roots of the lengths of the pendulums.*

(2) For the same pendulum *the times of oscillation are inversely as the square roots of the force of gravity at different places.*

---

* If the initial inclination is 5°, the second term of (4) is only 0.000476; if 1° the second term is only 0.000019.

(3) For the same time *the lengths of pendulums vary as the force of gravity.*

Hence by means of the pendulum the force of gravity at different places of the earth's surface may be determined. Let L be the length of a pendulum which vibrates seconds at the place where the value of g is to be found; then from (1) of Art. 194 we have

$$1 = \pi\sqrt{\frac{L}{g}}; \quad \therefore \quad g = \pi^2 L; \tag{1}$$

and from this formula g has been calculated at many places on the earth. The method of determining L accurately will be investigated in Chap. VII.

Cor.—If n be the number of vibrations performed during N seconds, and T the time of one vibration,

$$\text{then } n = \frac{N}{T}, \text{ by (1) of Art. 194 } = \frac{N}{\pi}\sqrt{\frac{g}{l}}. \tag{2}$$

Since gravity decreases according to a known law, as we ascend above the earth's surface, the comparison of the times of vibration of the same pendulum on the top of a mountain and at its base, would give approximately its height.

196. The Height of a Mountain Determined with the Pendulum.

—*A seconds pendulum is carried to the top of a mountain; required to find the height of the mountain by observing the change in the time of oscillation.*

Let r be the radius of the earth considered spherical; h the height of the mountain above the surface; l the length of the pendulum; g and g' the values of gravity on the earth's surface, and at the top of the mountain respectively. Then (Art. 174) we have

$$\frac{g}{g'} = \left(\frac{r+h}{r}\right)^2; \quad \therefore \quad g' = \frac{gr^2}{(r+h)^2}; \tag{1}$$

which is the force of gravity at the top of the mountain.

Let n = the number of oscillations which the seconds pendulum at the top of the mountain makes in 24 hours; then the time of oscillation $= \frac{24 \times 60 \times 60}{n}$. Hence from (1) of Art. 195, we have

$$\frac{24 \times 60 \times 60}{n} = \pi\sqrt{\frac{l}{g'}} = \pi\frac{r+h}{r}\sqrt{\frac{l}{g}}, \text{ by (1)};$$

$$\therefore \quad \frac{h}{r} = \frac{24 \times 60 \times 60}{n} - 1, \left(\text{since } \pi\sqrt{\frac{l}{g}} = 1\right), \tag{2}$$

which gives the height of the mountain in terms of the radius of the earth. For the sake of an example, suppose the pendulum to lose 5 seconds in a day; that is, to make 5 oscillations less than it would make on the surface of the earth.

Then $\qquad n = 24 \times 60 \times 60 - 5;$

which in (2) gives

$$\frac{h}{r} = \frac{24 \times 60 \times 60}{24 \times 60 \times 60 - 5} - 1$$

$$= \left(1 - \frac{1}{24 \times 60 \times 12}\right)^{-1} - 1 = \frac{1}{24 \times 60 \times 12} \text{ nearly};$$

$$\therefore \quad h = \frac{4000}{24 \times 60 \times 12} = \tfrac{1}{4} \text{ mile, nearly},$$

r being 4000 miles (approximately).

197. The Depth of a Mine Determined by Observing the Change of Oscillation in a Seconds Pendulum.—Let r be the radius of the earth as in the last case; h the depth of the mine; g and g' the values of gravity on the earth's surface and at the bottom of the mine. Then (Art. 171) we have

$$\frac{g}{g'} = \frac{r}{r-h}. \tag{1}$$

Let $n =$ the number of oscillations which the seconds pendulum at the bottom of the mine makes in 24 hours.

Then
$$\frac{24 \times 60 \times 60}{n} = \pi\sqrt{\frac{lr}{g(r-h)}}$$
$$= \sqrt{\frac{r}{r-h}}.$$
$$\therefore\ 1 - \frac{h}{r} = \left(\frac{n}{24 \times 60 \times 60}\right)^2,$$

from which h can be found. If, as before, the pendulum loses 5 seconds a day, we have

$$\frac{h}{r} = 1 - \left(1 - \frac{1}{24 \times 60 \times 12}\right)^2$$
$$= \frac{1}{12 \times 60 \times 12} \text{ nearly,}$$
$$\therefore\ h = \tfrac{1}{2} \text{ mile nearly.}$$

(See Price's Anal. Mech's, Vol. I, p. 590, also Pratt's Mech's, p. 376.)

198. Centripetal and Centrifugal Forces.—Since the pressure $\frac{v^2}{\rho}$, at any point, depends entirely upon the velocity at that point and the radius of curvature, it would remain the same if the forces X and Y were both zero, in which case it would be the whole normal pressure, R, against the curve. It is easily seen, therefore, that this pressure arises entirely from the inertia of the moving particle, *i. e.*, from its tendency at any point, to move in the direction of a tangent; and this tendency to motion along the tangent necessarily causes it to exert a pressure against the deflecting curve, and which requires the curve to oppose the resistance $\frac{v^2}{\rho}\cdot$ Hence, since the particle if left to itself, or if left to the action of a force along the tangent, would, by the law of inertia, continue to move along that tangent, $\frac{v^2}{\rho}$ is the effect of the force which deflects the particle from its otherwise rectilinear path, and draws it towards the centre of curvature. This force is called the *Centripetal Force*, which, therefore, may be defined to be *the force which deflects a particle from its otherwise rectilinear path.* The equal and opposite reaction exerted away from the centre is called *the Centrifugal Force*, which may be defined to be *the resistance which the inertia of a particle in motion opposes to whatever deflects it from its rectilinear path.* Centripetal and centrifugal are therefore the same quantity under different aspects. The action of the former is *towards* the centre of curvature, while that of the latter is *from* the centre of curvature. The two are called *central forces.* They determine the *direction* of motion of the particle but do not affect the *velocity*, since they act continually at right angles to its path. If a particle, attached to a string, be whirled about a centre, the intensity of these central forces is measured by the tension of the string. If the string be cut, the particle will move along a tangent to the curve with unchanged velocity.

COR. 1.—If m be the mass moving with velocity v, its centrifugal force is $m\frac{v^2}{\rho}\cdot$ If ω be the angular velocity described by the radius of curvature, then (Art. 160, Ex. 1), $v = \rho\omega$, and consequently

$$\text{the centrifugal force of } m = m\omega^2\rho. \qquad (1)$$

Cor. 2.—Let m move in a circle with a constant velocity, v; let $a =$ the radius of the circle, and T the time of a complete revolution; then $2\pi a = vT$;

$$\therefore \quad \text{the centrifugal force of } m = m\frac{4\pi^2 a}{T^2}; \qquad (2)$$

and thus *the centrifugal force in a circle varies directly as the radius of the circle, and inversely as the square of the periodic time.*

Cor. 3.—If m moves in the circle with a constant angular velocity, ω, then (Art. 160, Ex. 1), $v = a\omega$;

$$\therefore \quad \text{the centrifugal force of } m = m\,\omega^2 a; \qquad (3)$$

and therefore *varies directly as the radius of the circle.*

Thus if a particle of mass m is fastened by a string of length a to a point in a horizontal plane, and describes a circle in the plane about the given point as centre, the centrifugal force produces a tension of the string, and if ω is the constant angular velocity, the tension $= m\,\omega^2 a$.

199. The Centrifugal Force at the Equator.—Let R denote the equatorial radius of the earth $= 20926202$* feet, T the time of revolution upon its axis $= 86164$ seconds, and $\pi = 3.1415926$. Substituting these values in (2) of Art. 198, and denoting the centrifugal force at the equator by f, and the mass by unity, we have

$$f = \frac{4\pi^2 R}{T^2} = 0.11126 \text{ feet.} \qquad (1)$$

---

* Ency. Brit., Art. Geodesy.

The force of gravity at the equator has been found to be 32.09022: if this force were not diminished by the centrifugal force; *i. e.*, if the earth did not revolve on its axis the force of gravity at the equator would be

$$G = 32.09022 + 0.11126 = 32.20148 \text{ feet.} \qquad (2)$$

To determine the relation between the centrifugal force and the force of gravity, we divide (1) by (2) which gives

$$\frac{f}{G} = \frac{0.11126}{32.20148} = \frac{1}{289}, \text{ nearly.} \qquad (3)$$

that is, *the centrifugal force at the equator is* $\frac{1}{289}$ *of that which the force of gravity at the equator would be if the earth did not rotate.*

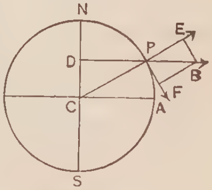

Fig. 85

200. Centrifugal Force at Different Latitudes on the Earth.—Let P be any particle on the earth's surface describing a circumference about the axis, NS, with the radius PD. Let $\phi = ACP =$ the latitude of P; R the radius, AC, of the earth; and R' the radius PD of the parallel of latitude passing through P. Then we have

$$R' = R \cos \phi. \qquad (1)$$

Let the centrifugal force at the point P, which is exerted in the direction of the radius DP, be represented by the line PB. Resolve this into the two components PF, acting along the tangent, and PE, acting along the normal. Then by (2) of Art. 198 we have

$$PB = \frac{4\pi^2 R'}{T^2}$$

$$= \frac{4\pi^2 R \cos \phi}{T^2}, \text{ by (1).} \qquad (2)$$

Hence, *the centrifugal force at any point on the earth's surface varies directly as the cosine of the latitude of the place.*

For the normal component we have

$$PE = PB \cos \phi$$

$$= \frac{4\pi^2 R \cos^2 \phi}{T^2} \text{ by (2)}$$

$$= f \cos^2 \phi, \text{ by (1) of Art. 199.} \quad (3)$$

Hence, *the component of the centrifugal force which directly opposes the force of gravity, at any point on the earth's surface, is equal to the centrifugal force at the equator, multiplied by the square of the cosine of the latitude of the place.*

Also
$$PF = PB \sin \phi$$

$$= \frac{4\pi^2 R \sin \phi \cos \phi}{T^2}, \text{ by (2)}$$

$$= \frac{f}{2} \sin 2\phi, \text{ by (1) of Art. 199 ;} \quad (4)$$

that is, *the component of the centrifugal force which tends to draw particles from any parallel of latitude,* P, *towards the equator, and to cause the earth to assume the figure of an oblate spheroid, varies as the sine of twice the latitude.*

The preceding calculation is made on the hypothesis that the earth is a perfect sphere, whereas it is an oblate spheroid ; and the attraction of the earth on particles at its surface decreases as we pass from the poles to the equator. The pendulum furnishes the most accurate method of determining the force of gravity at different places on the earth's surface.

201. The Conical Pendulum.—The Governor.—Suppose a particle, P, of mass m, to be attached to one end of a string of length l, the other end of which is fixed at A. The particle is made to describe a horizontal circle of radius PO, with uniform velocity round the vertical axis AO, so that it makes n revolutions per second. It is required to find the inclination, θ, of the string to the vertical, and the tension of the string.

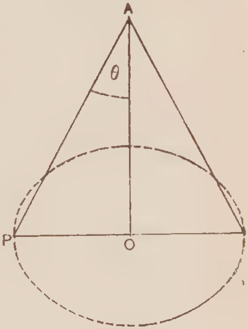

Fig. 86

The velocity of P in feet per second $= 2\pi n \cdot OP = 2\pi n l \sin\theta$. The forces acting upon it are the tension, T, of the string, the weight, m, of the particle, and the centrifugal force, $m\dfrac{4\pi^2 n^2 l^2 \sin^2\theta}{l \sin\theta}$ (Art. 198). Hence resolving, we have

$$\text{for horizontal forces,} \quad T \sin\theta = m \cdot 4\pi^2 n^2 l \sin\theta; \qquad (1)$$

$$\text{for vertical forces,} \quad T\cos\theta = mg. \qquad (2)$$

From (1)
$$T = m \cdot 4\pi^2 n^2 l, \qquad (3)$$

which in (2) gives

$$\cos\theta = \frac{g}{4\pi^2 n^2 l}, \qquad (4)$$

where T and θ are completely determined.

If the string be replaced by a rigid rod, which can turn about A in a ball and socket joint, the instrument is called a *conical pendulum*, and occurs in the *governor* of the steam-engine.

EXAMPLES.

1. If the length of the seconds pendulum be 39.1393 inches in London, find the value of g to three places of decimals. *Ans.* 32.191 feet.

2. In what time will a pendulum vibrate whose length is 15 inches? *Ans.* 0.62 sec. nearly.

3. In what time will a pendulum vibrate, whose length is double that of a seconds pendulum? *Ans.* 1.41 secs.

4. How many vibrations will a pendulum 3 feet long make in a minute? *Ans.* 62.55.

5. A pendulum which beats seconds, is taken to the top of a mountain one mile high; it is required to find the number of seconds which it will lose in 12 hours, allowing the radius of the earth to be 4000 miles. *Ans.* 10.8 secs.

6. What is the length of a pendulum to beat seconds at the place where a body falls $16\frac{1}{12}$ ft. in the first second? *Ans.* 39.11 ins. nearly.

7. If 39.11 ins. be taken as the length of the seconds pendulum, how long must a pendulum be to beat 10 times in a minute? *Ans.* $117\frac{1}{3}$ ft.

8. A particle slides down the arc of a circle to the lowest point; find the velocity at the lowest point, if the angle described round the centre is 60°. *Ans.* $\sqrt{gr}$.

9. A pendulum which oscillates in a second at one place, is carried to another place where it makes 120 more oscillations in a day; compare the force of gravity at the latter place with that at the former. *Ans.* $(\frac{8652}{8640})^2$.

10. Find the number of vibrations, n_1, which a pendulum will gain in N seconds by shortening the length of the pendulum.

Let the length, l, be decreased by a small quantity, l_1, and let n be increased by n_1; then from (2) of Art. 195 we get

$$n + n_1 = \frac{N}{\pi}\sqrt{\frac{g}{l - l_1}};$$

which, divided by (2) of Art. 195, gives

$$\frac{n + n_1}{n} = \left(\frac{l}{l - l_1}\right)^{\frac{1}{2}} = \left(1 - \frac{l_1}{l}\right)^{-\frac{1}{2}} = 1 + \frac{l_1}{2l} \text{ nearly.}$$

Hence $$n_1 = \frac{nl_1}{2l}.$$

11. If a pendulum be 45 inches long, how many vibrations will it gain in one day if the bob* be screwed up one turn, the screw having 32 threads to the inch?
Ans. 28.

12. If a clock loses two minutes a day, how many turns to the right hand must we give the nut in order to correct its error, supposing the screw to have 50 threads to the inch? *Ans.* 5·4 turns.

13. A mean solar day contains 24 hours, 3 minutes, 56·5 seconds, sidereal time; calculate the length of the pendulum of a clock beating sidereal seconds in London. See Ex. 1. *Ans.* 38·925 inches.

14. A heavy ball, suspended by a fine wire, vibrates in a small arc; 48 vibrations are counted in 3 minutes. Calculate the length of the wire. *Ans.* 45·87 feet.

15. The height of the cupola of St. Paul's, above the floor, is 340 ft.; calculate the number of vibrations a heavy body would make in half an hour, if suspended from the dome by a fine wire which reaches to within 6 inches of the floor. *Ans.* 176·4.

---

* The lower extremity of the pendulum.

16. A seconds pendulum is carried to the top of a mountain m miles high; assuming that the force of gravity varies inversely as the square of the distance from the centre of the earth, find the time of an oscillation.

Ans. $\left(\dfrac{4000 + m}{4000}\right)$ secs.

17. Prove that the lengths of pendulums vibrating during the same time at the same place are inversely as the squares of the number of oscillations.

18. In a series of experiments made at Harton coal-pit, a pendulum which beat seconds at the surface, gained $2\frac{1}{4}$ beats in a day at a depth of 1260 ft.; if g and g' be the force of gravity at the surface and at the depth mentioned, show that

$$\frac{g' - g}{g} = \tfrac{1}{19200}.$$

19. A pendulum is found to make 640 vibrations at the equator in the same time that it makes 641 at Greenwich; if a string hanging vertically can just sustain 80 lbs. at Greenwich, how many lbs. can the same string sustain at the equator? *Ans.* $80\frac{1}{4}$ lbs. about.

20. Find the time of descent of a particle down the arc of a cycloid, the axis of the cycloid being vertical and vertex downward; and show that the time of descent to the lowest point is the same whatever point of the curve the particle starts from.

Ans. $\pi\sqrt{\dfrac{r}{g}}$.

21. If in Ex. 20 the particle begins to move from the extremity of the base of the cycloid find the pressure at the lowest point of the curve.

Ans. $2g$; *i. e.*, the pressure is twice the weight of the particle.

22. Find the pressure on the lowest point of the curve in Art. 193, (1) when the particle starts from rest at the highest point, A, (Fig. 84), (2) when it starts from rest at the point B.

Ans. (1) $5g$; (2) $3g$; *i. e.*, (1) the pressure is five times the weight of the particle and (2) it is three times the weight of the particle.

23. In the simple pendulum find the point at which the tension on the string is the same as when the particle hangs at rest.

Ans. $y = \frac{2}{3}h$, where h is the height from which the pendulum has fallen.

24. If a particle be compelled to move in a circle with a velocity of 300 yards per minute, the radius of the circle being 16 ft., find the centrifugal force.

Ans. 14·06 ft. per sec.

25. If a body, weighing 17 tons, move on the circumference of a circle, whose radius is 1110 ft., with a velocity of 16 ft. per sec., find the centrifugal force in tons (take $g = 32 \cdot 1948$). *Ans.* 0·1217 ton.

26. If a body, weighing 1000 lbs., be constrained to move in a circle, whose radius is 100 ft., by means of a string capable of sustaining a strain not exceeding 450 lbs., find the velocity at the moment the string breaks.

Ans. 38·06 ft. per sec.

27. If a railway carriage, weighing 7·21 tons, moving at the rate of 30 miles per hour, describe a portion of a circle whose radius is 460 yards, find its centrifugal force in tons.

Ans. 0·314 ton.

28. If the centrifugal force, in a circle of 100 ft. radius, be 146 ft. per sec., find the periodic time.

Ans. 5·2 secs.

29. If the centrifugal force be 131 ozs., and the radius of the circle 100 ft., the periodic time being one hour, find the weight of the body. *Ans.* 386·309 tons.

30. Find the force towards the centre required to make a body move uniformly in a circle whose radius is 5 ft., with such a velocity as to complete a revolution in 5 secs.

Ans. $\frac{4\pi^2}{5}$.

31. A stone of one lb. weight is whirled round horizontally by a string two yards long having one end fixed; find the time of revolution when the tension of the string is 3 lbs.

Ans. $2\pi\sqrt{\frac{2}{g}}$ secs.

32. A weight, w, is placed on a horizontal bar, OA, which is made to revolve round a vertical axis at O, with the angular velocity ω; it is required to determine the position, A, of the weight, when it is upon the point of sliding, the coefficient of friction being f.

Ans. $\text{OA} = \frac{fg}{\omega^2}$.

33. Find the diminution of gravity at the Sun's equator caused by the centrifugal force, the radius of the Sun being 441000 miles, and the time of revolution on his axis being 607 h. 48 m. *Ans.* 0·0192 ft. per sec.

34. Find the centrifugal force at the equator of Mercury, the radius being 1570 miles, and the time of revolution 24 h. 5 m. *Ans.* 0·0435 ft. per sec.

35. Find the centrifugal force at the equator, (1) of Venus, radius being 3900 miles and time of revolution 23 h. 21 m., (2) of Mars, radius being 2050 miles and periodic time 24 h. 37 m., (3) of Jupiter, radius being 43500 miles and periodic time 9 h. 56 m., and (4) of Saturn, radius being 39580 miles and periodic time 10 h. 29 m.

Ans. (1) 0·11504 ft. per sec.; (2) 0·0544 ft. per sec.; (3) 7·0907 ft. per sec.; (4) 5·7924 ft. per sec.

36. Find the effect of centrifugal force in diminishing gravity in the latitude of 60°. [See (3) of Art. 200).

Ans. 0·028 ft. per sec.

37. Find (1) the diminution of gravity caused by centrifugal force, and (2) the component which urges particles towards the equator, at the latitude of 23°.

Ans. (1) 0·09 ft. per sec.; (2) 0·04 ft. per sec.

38. A railway carriage, weighing 12 tons, is moving along a circle of radius 720 yards, at the rate of 32 miles an hour; find the horizontal pressure on the rails.

Ans. 0·38 ton, nearly.

39. A railway train is going smoothly along a curve of 500 yards radius at the rate of 30 miles an hour; find at what angle a plumb-line hanging in one of the carriages will be inclined to the vertical. *Ans.* 2° 18′ nearly.

40. The attractive force of a mountain horizontally is f, and the force of gravity is g; show that the time of vibration of a pendulum will be $\pi\sqrt[4]{\dfrac{a^2}{g^2+f^2}}$; a being the length of the pendulum.

41. In motion of a particle down a cycloid prove that the vertical velocity is greatest when it has completed half its vertical descent.

42. When a particle falls from the highest to the lowest point of a cycloid show that it describes half the path in two-thirds of the time.

43. A railway train is moving smoothly along a curve at the rate of 60 miles an hour, and in one of the carriages a pendulum, which would ordinarily oscillate seconds, is observed to oscillate 121 times in two minutes. Show that the radius of the curve is very nearly a quarter of a mile.

44. One end of a string is fixed; to the other end a particle is attached which describes a horizontal circle with uniform velocity so that the string is always inclined at an angle of 60° to the vertical; show that the velocity of the particle is that which would be acquired in falling freely from rest through a space equal to three-fourths of the length of the string.

45. The horizontal attraction of a mountain on a particle at a certain place is such as would produce in it an acceleration denoted by $\frac{g}{n}$. Show that a seconds pendulum at that place will gain $\frac{21600}{n^2}$ beats in a day, very nearly.

46. In Art. 201, suppose l equal to 2 ft. and m to be 20 lbs., and that the system makes 10 revolutions per sec., and $g = 32$; find θ and T.

Ans. $\theta = \cos^{-1}\frac{1}{25\pi^2}$; $T = 500\pi^2$ pounds.

47. A tube, bent into the form of a plane curve, revolves with a given angular velocity, about its vertical axis; it is required to determine the form of the tube, when a heavy particle placed in it remains at rest in all parts of the tube.

(Take the vertical axis for the axis of y, and the axis of x horizontal, and let $\omega =$ the constant angular velocity). *Ans.* $x^2\omega^2 = 2gy$, if $x = 0$ when $y = 0$, *i. e.*, the curve is a parabola whose axis is vertical and vertex downwards.

48. A particle moves in a smooth straight tube which revolves with constant angular velocity round a vertical axis to which it is perpendicular, to determine the curve traced by the particle.

Let $\omega =$ the constant angular velocity; and (r, θ) the position of the particle at the time t, and let $r = a$ when

$t = 0$. Then since the motion of the particle is due entirely to the centrifugal force, we have

$$\frac{d^2r}{dt^2} = \omega^2 r\,; \quad \frac{dr^2}{dt^2} = \omega^2 (r^2 - a^2)$$

if $\dfrac{dr}{dt} = 0$, when $r = a$. Hence we have

$$r = \frac{a}{2}\left(e^{\omega t} + e^{-\omega t}\right).$$

CHAPTER IV

IMPACT

202. An Impulsive Force.—Hitherto we have considered force only as *continuous, i. e.*, as acting through a definite and finite portion of time, and producing a finite change of velocity in that time. Such a force is measured at any instant by the mass on which it acts multiplied by the acceleration which it causes. If a particle of mass m be moving with a velocity v, and be retarded by a constant force which brings it to rest in the time t, then the measure of this force is $\frac{mv}{t}$ (Art. 20). Now suppose the *time* t during which the particle is brought to rest to be made very *small;* then the *force* required to bring it to rest must be very *large;* and if we suppose t so small that we are unable to measure it, then the force becomes so great that we are unable to obtain its measure. A typical case is the blow of a hammer. Here the time during which there is contact is apparently infinitesimal, certainly too small to be measured by any ordinary methods; yet the effect produced is considerable. Similarly when a cricket ball is driven back by a blow from a bat, the original velocity of the ball is destroyed and a new velocity generated. Also when a bullet is discharged from a gun, a large velocity is generated in an extremely brief time. Forces acting in this way are called *impulsive forces. An impulsive force may therefore be defined to be a force which produces a finite change of motion in an indefinitely brief time. An Impulse is the effect of a blow.*

In such cases as these it is impossible accurately to determine the force and time; but we can determine their product, or Pt, since this is merely the change in velocity caused by the blow (Art. 20). Hence, in the case of blows, or impulsive forces, we do not attempt to measure the force and the time of action separately, but simply take the *whole momentum produced or destroyed, as the measure of the impulse.* Because impulsive forces produce their *effects* in an indefinitely short time they are sometimes called *instantaneous forces, i. e.*, forces requiring no time for their action. But no such force exists in nature; every force requires *time* for its action. There is no case in nature in which a finite change of motion is produced in an infinitesimal of time; for, whenever a finite velocity is generated or destroyed, a finite time is occupied in the process, though we may be unable to measure it, even approximately.

203. Impact or Collision.—When two bodies in relative motion come into contact with each other, an *impact* or *collision* is said to take place, and pressure begins to act between them to prevent any of their parts from jointly occupying the same space. This force increases from zero, when the collision begins, up to a very large magnitude at the instant of greatest compression. If, as is always the case in nature, each body possesses some degree of elasticity, and if they are not kept together after the impact by cohesion or by some artificial means, the mutual pressure between them, after reaching a maximum, will gradually diminish to zero. The whole process would occupy not greatly more or less than an hour if the bodies were of such dimensions as the earth, and such degrees of rigidity as copper, steel, or glass. In the case, however, of globes of these substances not exceeding a yard in diameter, the whole process is probably finished within a thousandth of a second.*

The impulsive forces are so much more intense than the ordinary forces, that during the brief time in which the former act, an ordinary force does not produce an effect comparable in amount with that produced by an impulsive force. For example, an impulsive force might generate a velocity of 1000 in less time than one-tenth of a second, while gravity in one-tenth of a second would generate a velocity of about three. Hence, in dealing with the effects of impulses, finite forces need not be considered.

---

* Thomson and Tait's Nat. Phil., p. 274.

204. Direct and Central Impact.—When two bodies impinge on each other, so that their centres before impact are moving in the same straight line, and the common tangent at the point of contact is perpendicular to the line of motion, the impact is said to be *direct* and *central.* When these conditions are not fulfilled, the impact is said to be *oblique.*

When two bodies impinge directly, one upon the other, the mutual action between them, at any instant, must be in the line joining their centres; and by the third law (Art. 166), it must be equal in amount on the two bodies. Hence, by Law II, they must experience equal changes of motion in contrary directions.

We may consider the impact as consisting of two parts; during the first part the bodies are coming into closer contact with each other, mutually displacing the particles in the vicinity of the point of contact, producing a compression and distortion about that point, which increases till it reaches a maximum, when the molecular reactions, thus called into play, are sufficient to resist further compression and distortion. At this instant it is evident that the points in contact are moving with the same velocity. No body in nature is perfectly *inelastic;* and hence, at the instant of greatest compression, the *elastic forces of restitution* are brought into action; and during the second part of the impact the mutual pressure, produced by the elastic forces, which were brought into action by the compression during the first part of the impact, tend to separate the two bodies, and to restore them to their original form.

205. Elasticity of Bodies.—Coefficient of Restitution.—It appears from experiment that bodies may be compressed in various degrees, and recover more or less their original forms after the compressing force has ceased; this property is termed *elasticity*. The force urging the approach of bodies is called the *force of compression;* the force causing the bodies to separate again is called the *force of restitution.* Elastic bodies are such as regain a part or all of their original form when the compressing force is removed. The ratio of the force of restitution to that of compression is called the *Coefficient of Restitution.** It has been found that this ratio, in the same bodies, is constant whatever may be their velocities.

When this ratio is unity the two forces are equal, and the body is said to be *perfectly elastic;* when the ratio is zero, or the force of restitution is nothing, the body is said to be *non-elastic;* when the ratio is greater than zero and less than unity, the body is said to be *imperfectly elastic.* There are no bodies either perfectly elastic or perfectly non-elastic, all being more or less elastic.

In the cases discussed the bodies will be supposed spherical, and in the case of direct impact of smooth spheres it is evident that they may be considered as particles, since they are symmetrical with respect to the line joining their centres.

The theory of the impact of bodies is chiefly due to Newton, who found, in his experiments, that, provided the impact is not so violent as to make any sensible indentation in either body, the relative velocity of separation after the impact bears a ratio to the relative velocity of approach before the impact, which is constant for the same two bodies. In Newton's experiments, however, the two bodies seem always to have been formed of the same substance. He found that the value of this ratio (the *coefficient of restitution*), for balls of compressed wool was about $\frac{5}{9}$, steel about the same, cork a little less, ivory $\frac{8}{9}$, glass $\frac{15}{16}$. The results of more recent experiments, made by Mr. Hodgkinson, and recorded in the *Report of the British Association for 1834*, show that the theory may be received as satisfactory, with the exception that the value of the ratio, instead of being quite constant, diminishes when the velocities are very large.

---

* Sometimes called Coefficient of Elasticity. Todhunter's Mech., p. 272.

206. Direct Impact of Inelastic Bodies.—*A sphere of mass* M, *moving with a velocity* v, *overtakes and impinges directly on another sphere of mass* M′, *moving in the same direction with velocity* v′, *and at the instant of greatest mutual compression the spheres are moving with a common velocity* V. *Determine the motion after impact, and the impulse during the compression.*

Let R denote the impulse during the compression, which acts on each body in opposite directions; and let us suppose the bodies to be moving from left to right. Then, since the impulse is measured by the amount of momentum gained by one of the impinging bodies or lost by the other (Art. 202), we have

$$\text{Momentum lost by } M = M(v - V) = R, \qquad (1)$$

$$\text{" gained by } M' = M'(V - v') = R, \qquad (2)$$

$$\therefore \quad M(v - V) = M'(V - v'). \qquad (3)$$

Solving (3) for V we get

$$V = \frac{Mv + M'v'}{M + M'}, \qquad (4)$$

which in (1) or (2) gives

$$R = \frac{MM'(v - v')}{M + M'}. \tag{5}$$

Hence *the common velocities of the two bodies after impact is equal to the algebraic sum of their momenta, divided by the sum of their masses*, and also, from (4), *the whole momentum after impact is equal to the sum of the momenta before.*

Cor. 1.—Had the balls been moving in opposite directions, for example had M' been moving from right to left, v' would have been negative, in which case we would have

$$V = \frac{Mv - M'v'}{M + M'}; \quad \text{and} \quad R = \frac{MM'(v + v')}{M + M'}. \tag{6}$$

From the first of these it follows that both balls will be reduced to rest if

$$Mv = M'v';$$

that is, if before impact they have equal and opposite momenta.

Cor. 2.—If M' is at rest before impact, $v' = 0$, and (4) becomes

$$V = \frac{Mv}{M + M'}. \tag{7}$$

If the masses are equal we have from (4) and (6)

$$V = \frac{v + v'}{2}, \quad \text{or} \quad \frac{v - v'}{2}, \tag{8}$$

according as they move in the same or in opposite directions.

207. Direct Impact of Elastic Bodies.—When the balls are elastic the problem is the same, up to the instant of greatest compression, as if they were inelastic; but at this instant, the force of restitution, or that tendency which elastic bodies have to regain their original form, begins to throw one ball forward with the same momentum that it throws the other back, and this mutual pressure is proportional to R (Art. 205).

Let e be the coefficient of restitution; then during the second part of the impact, an impulse, eR, acts on each ball in the same direction respectively as R acted during the compression. Let v_1 and v_1' be the velocities of the balls M and M' when they are finally separated. Then we have, as before,

$$\text{Momentum lost by } M = M(V - v_1) = eR, \qquad (1)$$

$$\text{" gained by } M' = M'(v_1' - V) = eR. \qquad (2)$$

From (1) we have

$$v_1 = V - \frac{eR}{M}$$

$$= \frac{Mv + M'v'}{M + M'} - \frac{eM'}{M + M'}(v - v')$$

by (4) and (5) of Art. 206,

$$= v - \frac{M'}{M + M'}(1 + e)(v - v'). \qquad (3)$$

Similarly from (2) we have

$$v_1' = v' + \frac{M}{M + M'}(1 + e)(v - v'); \qquad (4)$$

which are the velocities of the balls when finally separated.

These results may be more easily obtained by the consideration that the whole impulse is $(1 + e) R$; for this gives at once the whole momentum lost by M or gained by M' during compression and restitution as follows:

$$M(v - v_1) = (1 + e) R, \qquad (5)$$

and $$M'(v_1' - v') = (1 + e) R. \tag{6}$$

Substituting in (5) and (6) the value of R from (5) of Art. 206, we have the values of v and v_1' immediately.

Cor. 1.—If the balls are moving in opposite directions, v' becomes negative. If the balls are non-elastic, $e = 0$, and (3) and (4) reduce to (4) of Art. 206, as they should

Cor. 2.—If the balls are *perfectly elastic*, $e = 1$, and (3) and (4) become

$$v_1 = v - \frac{2M'}{M + M'}(v - v'), \tag{7}$$

$$v_1' = v' + \frac{2M}{M + M'}(v - v'). \tag{8}$$

Cor. 3.—Subtracting (4) from (3) and reducing, we get

$$\begin{aligned} v_1 - v_1' &= v - v' - (1 + e)(v - v'), \\ &= -e(v - v'). \end{aligned} \tag{9}$$

Hence, *the relative velocity after impact is* $-e$ *times the relative velocity before impact.*

Cor. 4.—Multiplying (3) and (4) by M and M', respectively, and adding, we get

$$Mv_1 + M'v_1' = Mv + M'v'. \tag{10}$$

Hence, as in Art. 206, *the algebraic sum of the momenta after impact is the same as before;* i. e., *there is no momentum lost*, which of course is a direct consequence of the third law of motion (Art. 169).

Cor. 5.—Suppose $v' = 0$, so that the body of mass M, moving with velocity v, impinges on a body of mass M' at rest, then (3) and (4) become

$$v_1 = \frac{M - eM'}{M + M'}\, v, \quad \text{and} \quad v_1' = \frac{M(1 + e)}{M + M'}\, v. \tag{11}$$

Hence the body which is struck goes onwards; and the striking body goes onwards, or stops, or goes backwards, according as M is greater than, equal to, or less than eM'. If $M' = eM$, then (11) becomes

$$v_1 = (1 - e)\, v, \quad \text{and} \quad v_1' = v. \tag{12}$$

Cor. 6.—If $M = M$ and $e = 1$; that is, if the balls are of equal mass, and *perfectly elastic*,* then (7) and (8) become, respectively,

$$v_1 = v', \quad \text{and} \quad v_1' = v; \tag{13}$$

that is, the balls interchange their velocities, and the motion is the same as if they had passed through one another without exerting any mutual action whatever.

Cor. 7.—If M' be infinite, and $v' = 0$, we have the case of a ball impinging directly upon a *fixed* surface; substituting these values in (3) it becomes

$$v_1 = -ev; \tag{14}$$

that is, *the ball rebounds from the fixed surface with a velocity* e *times that with which it impinged.*

208. Loss of Kinetic Energy† in the Impact of Bodies.—Squaring (9) of Art. 207, and multiplying it by MM', we have

$$MM'(v_1 - v_1')^2 = MM' e^2 (v - v')^2$$

$$= MM'(v - v')^2 - (1 - e^2)\, MM'(v - v')^2. \tag{1}$$

---

* This is the usual phraseology, but misleading, Ency. Brit., Vol. XV, Art. Mech's.

† See Art. 189.

Squaring (10) of Art. 207, we have

$$(Mv_1 + M'v_1')^2 = (Mv + M'v')^2. \tag{2}$$

Adding (1) and (2), we get

$$(M + M')(Mv_1^2 + M'v_1'^2) = (M + M')(Mv^2 + M'v'^2)$$
$$- (1 - e^2)MM'(v - v')^2;$$
$$\therefore\ \tfrac{1}{2}Mv_1^2 + \tfrac{1}{2}M'v_1'^2 = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}M'v'^2$$
$$- \tfrac{1}{2}(1-e^2)\frac{MM'}{M + M'}(v-v')^2, \tag{3}$$

the last term of which is the loss of kinetic energy by impact, since e can never be greater than unity. Hence, there is always a loss of kinetic energy by impact, except when $e = 1$, in which case the loss is zero; *i. e.*, when the coefficient of restitution is unity, no kinetic energy is lost. When $e = 0$ the loss is the greatest, and equal to

$$\tfrac{1}{2}\frac{MM'}{M + M'}(v - v')^2. \tag{4}$$

From (3) we see that during compression kinetic energy to the amount of $\frac{1}{2}\frac{MM'}{M + M'}(v - v')^2$ is lost; and then during restitution, e^2 times this amount is regained.

REM.—From the *theory of kinetic energy* it appears that, in every case in which energy is lost by resistance, heat is generated; and from Joule's* investigations we learn that the quantity of heat so generated is a perfectly definite *equivalent* for the energy lost; and also that, in any natural action, there is never a development of energy which cannot be accounted for by the disappearance of an equal amount elsewhere by means of some known physical agency. Hence, the kinetic energy which appears to be lost in the above cases of impact, is only transformed, partly into heating the bodies and the surrounding air, and partly into sonorous vibrations, as in the impact of a hammer on a bell.

---

* See "The Correlation and Conservation of Forces," by Helmholtz, Faraday, Liebig, etc.; also "Heat as a Mode of Motion," by Prof. Tyndall. Also B. Stewart's "Conservation of Energy."

209. Oblique Impact of Bodies.—The only other case which we shall treat of is that of oblique impact when the bodies are spherical and perfectly smooth.

A particle impinges with a given velocity, and in a given direction, on a smooth plane; required to determine the motion after impact.

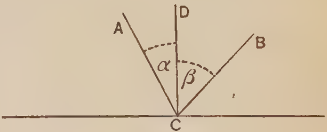

Fig. 87

Let AC represent the direction of the velocity before impact, meeting the plane at C, and CB the direction after impact. Draw CD perpendicular to the plane; then since the plane is smooth its impulsive reaction will be along CD.

Let v and v_1 denote the velocities before and after impact, respectively; and let α and β denote the angles ACD and BCD.

Resolve v along the plane and perpendicular to it. The former will not be altered, since the impulsive force acts perpendicular to the plane; the latter may be treated as in the case of direct impact, and will therefore, after impact, be e times what it was before (Art. 207, Cor. 7). Hence, resolving v_1 along, and perpendicular to the plane, we have

$$v_1 \sin \beta = v \sin \alpha, \tag{1}$$

$$v_1 \cos \beta = -e\, v \cos \alpha. \tag{2}$$

Dividing (2) by (1), we get

$$\cot \beta = - e \cot \alpha. \tag{3}$$

Squaring (1) and (2), and adding, we get

$$v_1^2 = v^2 (\sin^2 \alpha + e^2 \cos^2 \alpha). \tag{4}$$

Thus (3) determines the *direction*, and (4) the *magnitude* of the velocity after impact.

The angle ACD is called the *angle of incidence*, and the angle BCD *the angle of reflexion.*

Cor. 1.—If the elasticity be perfect, or $e = 1$, we have from (3) and (4),

$$\cot \beta = - \cot \alpha, \text{ or } \beta = - \alpha; \tag{5}$$

and

$$v_1^2 = v^2, \text{ or } v_1 = v. \tag{6}$$

Hence, *in perfectly elastic balls the angles of incidence and reflexion are numerically equal, and the velocities before and after impact are equal.* This is the ordinary rule in the case of a billiard ball striking the cushion.

Cor. 2.—Suppose $e = 0$; then from (3), $\beta = 90°$. Thus, if there is no elasticity, the body after impact moves along the plane with the velocity $v \sin \alpha$.

If $\alpha = 0$, so that the impact is *direct*, we have from (4), $v_1 = ev$; *i. e.*, after the impact the body rebounded along its former course with e times its former velocity.

If $\alpha = 0$, and $e = 0$, then from (4), $v_1 = 0$, and the body is brought to rest by the impact.

Sch.—Of course the results of this article are applicable to cases of impact on any smooth surface, by substituting for the plane on which the impact has been supposed to take place the plane which is tangent to the surface at the point of impact.

210. Oblique Impact of Two Smooth Spheres.—*Two smooth spheres, moving in given directions and with given velocities, impinge; to determine the impulse and the subsequent motion.*

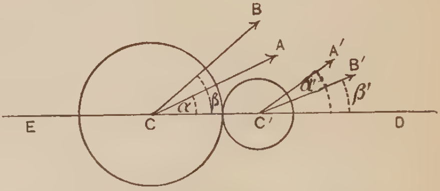

Fig. 88

Let the masses of the spheres be M, M'; their centres C, C′; their velocities before impact v and v', and after impact v_1 and v_1'. Let ED be the line which joins their centres at the instant of impact (called the line of impact): CA and CB the directions of motion of the impinging sphere, M, before and after impact; and C′A′ and C′B′ those of the other sphere; let α, α' be the angles, ACD and A′C′D, which the original directions of motion make with the line of impact; β, β' the angles, BCD and B′C′D, which their directions make after the impact.

It is evident that, since the spheres are smooth, the entire mutual impulsive pressure takes place in the line joining the centres at the instant of impact. Let R be the impulse, and e the coefficient of restitution. Resolve all the velocities along the line of impact and at right angles to it; the latter will not be affected by the impact, and the former will be affected exactly in the same way as if the impact had been direct. Hence, since the velocities in the line of impact are $v \cos \alpha$, $v' \cos \alpha'$, $v_1 \cos \beta$, $v_1' \cos \beta'$, we have, by substituting in (3) and (4) of Art. 207,

$$v_1 \cos \beta = v \cos \alpha - \frac{M'}{M+M'}(1+e)(v \cos \alpha - v' \cos \alpha'), \quad (1)$$

$$v_1' \cos\beta' = v' \cos\alpha' + \frac{M}{M+M'}(1+e)(v\cos\alpha - v'\cos\alpha'), \quad (2)$$

which are the final velocities of the two spheres along the line of impact ED.

Also, from (5) of Art. 206, we obtain by substitution,

$$R = \frac{MM'}{M+M'}(v \cos\alpha - v' \cos\alpha'), \quad (3)$$

(See Tait and Steele's Dynamics of a Particle, p. 323.)

Cor. 1.—Multiplying (1) by M, and (2) by M', and adding we get

$$Mv_1 \cos\beta + M'v_1' \cos\beta' = Mv\cos\alpha + M'v'\cos\alpha', \quad (4)$$

which shows that *the momentum of the system resolved along the line of impact is the same after impact as before.*

Cor. 2.—Subtracting (2) from (1) we obtain,

$$v_1 \cos\beta - v_1' \cos\beta' = -e(v\cos\alpha - v'\cos\alpha'). \quad (5)$$

That is, *the relative velocity, resolved along the line of impact, after impact is $-e$ times its value before.*

EXAMPLES.

1. A body* weighing 3 lbs. moving with a velocity of 10 ft. per second, impinges on a body weighing 2 lbs., and moving with a velocity of 3 ft. per second; find the common velocity after impact. *Ans.* $7\frac{1}{5}$ ft. per second.

2. A body weighing 7 lbs. moving 11 ft. per second, impinges on another at rest weighing 15 lbs.; find the common velocity after impact. *Ans.* $3\frac{1}{2}$ ft. per second.

---

* The bodies are inelastic unless otherwise stated. The first 27 examples are in *direct* impact.

3. A body weighing 4 lbs. moving 9 ft. per second, impinges on another body weighing 2 lbs. and moving in the opposite direction with a velocity of 5 ft. per second; find the common velocity after impact.

Ans. $4\frac{1}{3}$ ft. per second.

4. A body, M', weighing 5 lbs. moving 7 ft. per second, is impinged upon by a body, M, weighing 6 lbs. and moving in the same direction; after impact the velocity of M' is doubled: find the velocity of M before impact.

Ans. $19\frac{5}{6}$ ft. per second.

5. Two bodies, weighing 2 lbs., and 4 lbs., and moving in the same direction with the velocities of 6 and 9 ft. respectively, impinge upon each other; find their common velocity after impact. *Ans.* 8 ft. per second.

6. A weight of 2 lbs., moving with a velocity of 20 ft. per second, overtakes one of 5 lbs., moving with a velocity of 5 ft. per second; find the common velocity after impact.

Ans. $9\frac{2}{7}$ ft. per second.

7. If the same bodies *met* with the same velocities find the common velocity after impact.

Ans. $2\frac{1}{7}$ ft. per second in the direction of the first.

8. Two bodies of different masses, are moving towards each other, with velocities of 10 ft. and 12 ft. per second respectively, and continue to move after impact with a velocity of $1 \cdot 2$ ft. per second in the direction of the greater; compare their masses. *Ans.* As 3 to 2.

9. A body impinges on another of twice its mass at rest: show that the impinging body loses two-thirds of its velocity by the impact.

10. Two bodies of unequal masses moving in opposite directions with momenta numerically equal meet; show that the momenta are numerically equal after impact.

11. A body, M, weighing 10 lbs. moving 8 ft. per second, impinges on M', weighing 6 lbs. and moving in the same direction 5 ft. per second; find their velocities after impact, supposing $e = 1$.

Ans. Velocity of $M = 5\frac{3}{4}$; velocity of $M' = 8\frac{3}{4}$.

12. A body, M, weighing 4 lbs. moving 6 ft. per second, meets M' weighing 8 lbs. and moving 4 ft. per second; find their velocities after impact, $e = 1$.

Ans. Each body is reflected back, M with a velocity of $7\frac{1}{3}$ and M' with a velocity of $2\frac{2}{3}$.

13. Two balls, of 4 and 6 lbs. weight, impinge on each other when moving in the same direction with velocities of 9 and 10 ft. respectively; find their velocities after impact, supposing $e = \frac{4}{5}$. *Ans.* 10·08 and 9·28.

14. Find the kinetic energy lost by impact in example 5.

Ans. $\frac{36}{193}$.

15. Two bodies weighing 40 and 60 lbs. and moving in the same direction with velocities of 16 and 26 ft. respectively, impinge on each other: find the loss of kinetic energy by impact. *Ans.* 37·3.

16. An arrow shot from a bow starts off with a velocity of 120 ft. per second; with what velocity will an arrow twice as heavy leave the bow, if sent off with three times the force? *Ans.* 180 ft. per second.

17. Two balls, weighing 8 ozs. and 6 ozs. respectively, are simultaneously projected upwards, the former rises to a height of 324 ft. and the latter to 256 ft.; compare the forces of projection. *Ans.* As 3 to 2.

18. A freight train, weighing 200 tons, and traveling 20 miles per hr. runs into a passenger train of 50 tons, standing on the same track; find the velocity at which the remains of the passenger train will be propelled along the track, supposing $e = \frac{1}{5}$. *Ans.* 19·2 miles per hr.

19. There is a row of ten perfectly elastic bodies whose masses increase geometrically by the constant ratio 3, and the first impinges on the second with the velocity of 5 ft. per second; find the velocity of the last body.

Ans. $\frac{5}{512}$ ft. per second.

20. A body weighing 5 lbs. moving with a velocity of 14 ft. per second, impinges on a body weighing 3 lbs., and moving with a velocity of 8 ft. per second; find the velocities after impact supposing $e = \frac{1}{3}$. *Ans.* 11 and 13.

21. Two bodies are moving in the same direction with the velocities 7 and 5; and after impact their velocities are 5 and 6; find e, and the ratio of their masses.

Ans. $e = \frac{1}{2}$; $M' = 2M$.

22. A body weighing two lbs. impinges on a body weighing one lb.; e is $\frac{1}{2}$, show that $v_1 = \frac{1}{2}(v + v')$, and that $v_1' = v$.

23. Two bodies moving with numerically equal velocities in opposite directions, impinge on each other; the result is that one of them turns back with its original velocity, and the other follows it with half that velocity; show that one body is four times as heavy as the other, and that $e = \frac{1}{4}$.

24. A strikes B, which is at rest, and after impact the velocities are numerically equal; if r be the ratio of B's mass to A's mass, show that e is $\frac{2}{r - 1}$, and that B's mass is at least three times A's mass.

25. A body impinges on an equal body at rest; show that the kinetic energy before impact cannot be greater than twice the kinetic energy after impact.

26. A series of perfectly elastic balls are arranged in the same straight line; one of them impinges on the next, then this on the next and so on; show that if their masses form a geometric progression of which the common ratio

is r, their velocities after impact form a geometric progression of which the common ratio is $\frac{2}{r+1}$.

27. A ball falls from rest at a height of 20 ft. above a fixed horizontal plane; find the height to which it will rebound, e being $\frac{3}{4}$, and g being 32. *Ans.* $11\frac{1}{4}$ feet.

28. A ball impinges on an equal ball at rest, the elasticity being perfect; if the original direction of the striking ball is inclined at an angle of 45° to the straight line joining the centres, determine the angle between the directions of motion of the striking ball before and after impact. *Ans.* 45°.

29. A ball falls from a height h on a horizontal plane, and then rebounds; find the height to which it rises in its ascent. *Ans.* e^2h.

30. A ball of mass M, impinges on a ball of mass M', at rest; show that the tangent of the angle between the old and new directions of the motion of the impinging body is

$$\frac{1+e}{2} \cdot \frac{M' \sin 2\alpha}{M + M' (\sin^2 \alpha - e \cos^2 \alpha)}.$$

31. A ball of mass M impinges on a ball of mass M' at rest; find the condition in order that the directions of motion of the impinging ball before and after impact may be at right angles. *Ans.* $\tan^2 \alpha = \frac{M'e - M}{M' + M}$.

32. A ball impinges on an equal ball at rest, the angle between the old and new directions of motion of the impinging ball is 60°; find the velocity after impact, e being 1. *Ans.* $v \sin 30°$.

33. A ball impinges on an equal ball at rest, e being 1; find the condition under which the velocities will be equal after impact. *Ans.* $\alpha = 45°$

34. A ball is projected from the middle point of one side of a billiard table, so as to strike first an adjacent side, and then the middle point of the side opposite to that from which it started; find where the ball must hit the adjacent side, its length being b.

Ans. At the distance $\frac{b}{1+e}$ from the end nearest the opposite side.

CHAPTER V.

WORK AND ENERGY.

211. Definition and Measure of Work.—*Work is the production of motion against resistance.* A force is said to *do work*, if it moves the body to which it is applied; and the work done by it is measured by the product of the force into the space through which it moves the body (Art. 101, Rem.).

Thus, the work done in lifting a weight through a vertical distance is proportional to the weight lifted and the vertical distance through which it is lifted. *The unit of work* used in England and in this country *is that which is required to overcome the weight of a pound through the vertical height of a foot*, and is called *a foot-pound.* For instance, if a weight of 10 lbs. is raised to a height of 5 ft., or 5 lbs. raised to a height of 10 ft., 50 foot-pounds of work must have been expended in overcoming the resistance of gravity. Similarly, if it requires a force of 50 lbs. to move a load on a horizontal plane over a distance of 100 ft., 5000 foot-pounds of work must have been done. If a carpenter urges forward a plane through 3 ft. with a force of 12 lbs., he does 36 foot-pounds of work; or, if a weight of 7 lbs. descends through 10 ft., gravity does 70 foot-pounds of work on it.

Hence, the number of units of work, or foot-pounds, necessary to overcome a constant resistance of P pounds through a distance of S feet is equal to the product PS.

From this it appears that, if the point of application move always perpendicular to the direction in which the force acts, such a force does no work. Thus, no work is done by gravity in the case of a particle moving on a horizontal plane, and when a particle moves on any smooth surface no work is done by the force which the surface exerts upon it.

Neither *force* nor *motion* alone is sufficient to constitute *work;* so that a man who merely supports a load without moving it, does no work, in the sense in which that term is used mechanically, any more than a column does which sustains a heavy weight upon its summit.

If a body is moved in the direction *opposite* to that in which its weight acts, the agent raising it does work upon it, while the work done by the earth's attraction is *negative.* When the work done by a force is negative, *i. e.*, when the point of application moves in the direction opposite to that in which the force acts, this is frequently expressed by saying that work is done *against* the force. In the above case work is done *by* the force lifting the body, and *against* the earth's attraction.

212. General Case of Work done by a Force.—When either the magnitude or direction of a force varies, or if both of them vary, the work done by the force during any finite displacement cannot be defined as in Art. 211. In this case the work done during any indefinitely small displacement may be found by supposing the magnitude and direction of the force constant during the displacement, and finding the work done as in Art. 211; then taking the sum of all such elements of work done during the consecutive small displacements, which together make up the finite displacement, we obtain the whole work done by the force during such finite displacement.

Thus let a force, P, act at a point, O, in the direction OP (Fig. 50) and let us suppose the point, O, to move into any other position, A, very near O. If θ be the angle between the direction, OP, of the force and the direction, OA, of the displacement of the point of application, then the product, $P \cdot OA \cos\theta$, is called the work done by the force. If we drop a perpendicular, AN, on OP, the work done by the force is also equal to the product $P \cdot ON$, where ON is to be estimated as positive when in the direction of the force. If several forces act, the work done by each can be found in the same way; and the sum of all these is the work done by the whole system of forces.

It appears from this that the work done by any force during an infinitesimal displacement of the point of application, is the product of the resolved part of the force in the direction of the displacement into the displacement; and this is the same as the *virtual moment* of the force, which has been described in Art. 101. In Statics we are concerned only with the small *hypothetical* displacement which we give the point of application of the force in applying the principle of virtual velocities. But in Kinetics the bodies are in motion; the force *actually* displaces its point of application in such a manner that the displacement has a projection along the direction of the force. If ds denote the projection of any elementary arc of a curve along the direction of P, the work done by P in this displacement is Pds. The sum of all these elements of work done by P in its motion over a finite space is the whole work found by taking the integral of Pds between proper limits.

Hence generally, if s be an arc of the path of a particle, P the tangential component of the forces which act on it, the work done on the particle between any two points of its path is

$$\int Pds, \tag{1}$$

the integral being taken between limits corresponding to the initial and final positions of the particle.

213. Work on an Inclined Plane.—Let α be the inclination of the plane to the horizon, W the weight moved, s the distance along the plane through which the weight is moved. Resolve W into two components, one along the plane and the other perpendicular to it; the former, $W \sin \alpha$, is the component which resists motion along the plane. Hence the amount of work required to draw the weight up the plane $= W \sin \alpha \cdot s = W \times$ the vertical height of the plane; *i. e., the amount of work required is unchanged by the substitution of the oblique path for the vertical. Hence the work in moving a body up an inclined plane, without friction, is equal to the product of the weight of the body by the vertical height through which it is raised.*

Cor. 1.—If the plane be rough, let $\mu =$ the coefficient of friction; then since the normal component of the weight is $W \cos \alpha$, the resistance of friction is $\mu W \cos \alpha$ (Art. 92). The work required consists of two parts, (1) raising the weight along the plane, and (2) overcoming the resistance of friction along the plane, the former $= W \sin \alpha \cdot s$, and the latter is $\mu W \cos \alpha \cdot s$. Hence *the whole work necessary to move the weight up the plane is*

$$W (\sin \alpha + \mu \cos \alpha) s. \tag{1}$$

Since $s \sin \alpha$ represents the *vertical* height through which the weight is raised, and $s \cos \alpha$ the *horizontal* space through which it is drawn, this result may be stated thus: *The work expended is the same as that which would be required to raise the weight through the vertical height of the plane, together with that which would be required to draw the body along the base of the plane horizontally against friction.*

Cor. 2.—*If a body be dragged through a space,* s, *down an inclined plane, which is too rough for the body to slide down by itself, the work done is*

$$W (\mu \cos \alpha - \sin \alpha) s. \tag{2}$$

Cor. 3.—If $h =$ the height of the inclined plane, and $b =$ its horizontal base, then the work done against gravity to move the body up the plane $= Wh$; and the work done against friction to move the body along the plane, supposing it to be horizontal, $= \mu b W$. Hence (Cor. 1) the total work done is

$$Wh + \mu b W. \tag{3}$$

If the body be drawn down the plane, the total work expended (Cor. 2) is

$$-Wh + \mu b W. \tag{4}$$

If in (4) the former term is greater than the latter, gravity does more work than what is expended on friction, and the body slides down the plane with accelerated velocity.

SCH. 1.—If the inclination of the plane is small, as it is in most cases which occur in practice, as in common roads and railroads, $\cos \alpha$ may without any important error be taken as equal to unity, and the expression for the work becomes (Cors. 1 and 2)

$$W(\mu s \pm s \sin \alpha), \tag{5}$$

the upper or lower sign being taken according as the body is dragged *up* or *down* the plane.

SCH. 2.—If the inclination of the plane is small, as in the case of railway gradients, the pressure upon the plane will be very nearly equal to the weight of the body; and the total work in moving a body along an inclined plane will be from (3) and (4),

$$\mu l W \pm Wh, \tag{6}$$

where $\mu l W$ is the work due to friction along the plane of length l, and Wh is the work due to gravity, the proper sign being taken as in (5).

EXAMPLES.

1. How much work is done in lifting 150 and 200 lbs. through the heights of 80 and 120 ft. respectively.

$$\begin{aligned}\text{The work done} &= 150 \times 80 + 200 \times 120 \\ &= 36000 \text{ foot-pounds, } \textit{Ans.}\end{aligned}$$

2. A body weighing 500 lbs. slides on a rough horizontal plane, the coefficient of friction being 0.1; how much work must be done against friction to move the body over 100 ft.?

Here the friction is a force of 50 lbs. acting directly opposite to the motion; hence the work done against friction to move the body over 100 ft. is

$$50 \times 100 = 5000 \text{ foot-pounds, } \textit{Ans.}$$

3. A train weighs 100 tons; the total resistance is 8 lbs. per ton; how much work must be expended in raising it to the top of an inclined plane a mile long, the inclination of the plane being 1 vertical to 70 horizontal.

Here the work done against friction (Sch. 2)

$$= 800 \times 5280 = 4224000 \text{ foot-pounds,}$$

and the work done against gravity

$$= 224000^* \times 5280 \times \tfrac{1}{70} = 16896000 \text{ foot-pounds,}$$

so that the whole work = 21120000 foot-pounds.

4. A train weighing 100 tons moves 30 miles an hour along a horizontal road; the resistances are 8 lbs. per ton; find the quantity of work expended each hour.

Ans. 126720000 foot-pounds.

5. If 25 cubic feet of water are pumped every 5 minutes from a mine 140 fathoms deep, required the amount of work expended per minute, a cubic foot of water weighing 62½ lbs. *Ans.* 262500 foot-pounds.

6. How much work is done when an engine weighing 10 tons moves half a mile on a horizontal road, if the total resistance is 8 lbs. per ton.

Ans. 211200 foot-pounds.

7. If a weight of 1120 lbs. be lifted up by 20 men, 20 ft. high, twice in a minute, how much work does each man do per hour? *Ans.* 134400 foot-pounds.

---

* One ton being 2240 lbs. unless otherwise stated.

8. A body falls down the whole length of an inclined plane on which the coefficient of friction is 0.2. The height of the plane is 10 ft. and the base 30 ft. On reaching the bottom it rolls horizontally on a plane, having the same coefficient of friction. Find how far it will roll.

Ans. 20 ft.

9. How much work will be required to pump 8000 cubic feet of water from a mine whose depth is 500 fathoms.

Ans. 1500000000 foot-pounds.

10. A horse draws 150 lbs. out of a well, by means of a rope going over a fixed pulley, moving at the rate of $2\frac{1}{2}$ miles an hour; how many units of work does this horse perform a minute, neglecting friction.

Ans. 33000 units of work.

214. Horse Power.—It would be inconvenient to express the power of an engine in foot-pounds, since this unit is so small; the term *Horse Power* is therefore used in measuring the performance of steam engines. From experiments made by Boulton and Watt it was estimated that a horse could raise 33000 lbs. vertically through one foot in one minute. This estimate is probably too high on the average, but it is still retained. Whether it is greater or less than the power of a horse it matters little, while it is a power so well defined. A *Horse Power therefore means a power which can perform 33000 foot-pounds of work in a minute.* Thus, when we say that the *actual* horse power of an engine is ten, we mean that the engine is able to perform 330000 foot-pounds of work per minute.

It has been estimated that $\frac{2}{3}$ of the 33000 foot-pounds would be about the work of a horse of average strength. A mule will perform $\frac{2}{3}$ the work of a horse. An ass will perform about $\frac{1}{5}$ the work of a horse. A man will do about $\frac{1}{10}$ the work of a horse, or about 3300 units of work per minute. See Evers' Applied Mech's; also Byrne's Practical Mech's.

215. Work of Raising a System of Weights.—Let P, Q, R, be any three weights at the distances, p, q, r, respectively above a fixed horizontal plane. Then [Art. 59 (3)] or (Art. 73, Cor. 3), the distance of the centre of gravity of P, Q, R, above this fixed horizontal plane is

$$\frac{Pp + Qq + Rr}{P + Q + R}. \tag{1}$$

Now suppose that the weights are raised vertically through the heights a, b, c, respectively. Then the distance of the centre of gravity of the three weights, in the new position, above the same fixed horizontal plane is

$$\frac{P(p + a) + Q(q + b) + R(r + c)}{P + Q + R}. \tag{2}$$

Subtracting (1) from (2), we have

$$\frac{Pa + Qb + Rc}{P + Q + R}, \tag{3}$$

for the vertical distance between the two positions of the centre of gravity of the three bodies.

Now the work of raising vertically a weight equal to the sum of P, Q, R, through the space denoted by (3) is the product of the sum of the weights into the space, which is

$$Pa + Qb + Rc, \tag{4}$$

but (4) is the work of raising the three weights P, Q, R, through the heights a, b, c, respectively. In the same way this may be shown for any number of weights.

Hence when several weights are raised vertically through different heights, the whole work done is the same as that of raising a weight equal to the sum of the weights vertically from the first position of their centre of gravity to the last position. (See Todhunter's Mech's, p. 338.)

EXAMPLES.

1. How many horse-power would it take to raise 3 cwt. of coal a minute from a pit whose depth is 110 fathoms?

$$\text{Depth} = 110 \times 6 = 660 \text{ feet.}$$

$$3 \text{ cwt.} = 112 \times 3 = 336 \text{ lbs.}$$

Hence the work to be done in a minute

$$= 660 \times 336 = 221760 \text{ foot-pounds.}$$

Therefore the horse-power

$$= 221760 \div 33000 = 6.72, \textit{ Ans.}$$

2. Find how many cubic feet of water an engine of 40 horse-power will raise in an hour from a mine 80 fathoms deep, supposing a cubic foot of water to weigh 1000 ozs.

Work of the engine per hour $= 40 \times 33000 \times 60$ foot-pounds.

Work expended in raising one cubic foot of water through 80 fathoms $= \frac{1000}{16} \times 80 \times 6 = 30000$ foot-pounds.

Hence the number of cubic feet raised in an hour

$$= 40 \times 33000 \times 60 \div 30000 = 2640, \textit{ Ans.}$$

3. Find the horse-power of an engine which is to move at the rate of 20 miles an hour up an incline which rises 1 foot in 100, the weight of the engine and load being 60 tons, and the resistance from friction 12 lbs. per ton.

The horizontal space passed over in a minute $= 1760$ ft.; the vertical space is one-hundredth of this $= 17.60$ ft. Hence from (6) of Art. 213, we have

$$12 \times 1760 \times 60 + 60 \times 2240 \times 17.6 = 1760 \times 2064 \text{ foot-pounds.}$$

Therefore the horse-power

$$= 1760 \times 2064 \div 33000 = 110.08, \textit{ Ans.}$$

4. A well is to be dug 20 ft. deep, and 4 ft. in diameter: find the work in raising the material, supposing that a cubic foot of it weighs 140 lbs.

Here the weight of the material to be raised

$$= 4\pi \times 20 \times 140 = 140 \times 80\pi \text{ lbs.}$$

The work done is equivalent to raising this through the height of 10 ft. (Art. 215). Hence the whole work

$$= 140 \times 80\pi \times 10 = 112000\pi \text{ foot-pounds, } \textit{Ans.}$$

5. Find the horse-power of an engine that would raise T tons of coal per hour from a pit whose depth is a fathoms.

$$\text{Work per minute} = \frac{T \times 2240 \times a \times 6}{60} = 224aT;$$

$$\therefore \quad \text{the horse-power} = \frac{224aT}{33000}, \textit{ Ans.}$$

6. Required the work in raising water from three different levels whose depths are a, b, c fathoms respectively; from the first A, from the second B, from the third C, cubic feet of water are to be raised per minute.

Work in raising water from the first level

$$= 62.5\, A \times a \times 6 = 375\, A\cdot a;$$

and so on for the work in the other levels;

$$\therefore \quad \text{work per min.} = 375\,(A\cdot a + B\cdot b + C\cdot c) \text{ foot-pounds.}$$

7. Find the horse-power of an engine which draws a load of T tons along a level road at the rate of m miles

an hour, the friction being p pounds per ton, all other resistances being neglected.

Work of the engine per minute

$$= Tp\frac{5280\,m}{60} = 88\,Tpm.$$

$$\therefore\quad \text{H.-P.}^* = \frac{88\,Tpm}{33000} = \frac{8\,Tpm}{3000},\ \textit{Ans.}$$

8. Required the number of horse-power to raise 2200 cubic ft. of water an hour, from a mine whose depth is 63 fathoms. *Ans.* $26\frac{1}{4}$.

9. What weight of coal will an engine of 4 horse-power raise in one hour from a pit whose depth is 200 ft.? *Ans.* 39600 lbs.

10. In what time will an engine of 10 horse-power raise 5 tons of material from the depth of 132 ft.? *Ans.* 4·48 minutes.

11. How many cubic feet of water will an engine of 36 horse-power raise in an hour from a mine whose depth is 40 fathoms? *Ans.* 4752 cubic feet.

12. The piston of a steam engine is 15 ins. in diameter; its stroke is $2\frac{1}{2}$ ft. long; it makes 40 strokes per minute; the mean pressure of the steam on it is 15 lbs. per square inch; what number of foot-pounds is done by the steam per minute, and what is the horse-power of the engine? *Ans.* 265072·5 foot-pounds; 8·03 H.-P.

13. A weight of $1\frac{1}{2}$ tons is to be raised from a depth of 50 fathoms in one minute; determine the horse-power of the engine capable of doing the work. *Ans.* $30\frac{6}{11}$ H.-P.

---

[*] The letters H.-P. are often used as abbreviations of the words horse-power.

14. The resistance to the motion of a certain body is 440 lbs.; how many foot-pounds must be expended in making this body move over 30 miles in one hour? What must be the horse-power of an engine that does the same number of foot-pounds in the same time?

Ans. 69696000 foot-pounds; $35\frac{1}{5}$ H.-P.

15. An engine draws a load of 60 tons at the rate of 20 miles an hour; the resistances are at the rate of 8 lbs. per ton; find the horse-power of the engine. *Ans.* 25·6.

16. How many cubic feet of water will an engine of 250 horse-power raise per minute from a depth of 200 fathoms?

Ans. 110 cubic ft.

17. There is a mine with three shafts which are respectively 300, 450, and 500 ft. deep; it is required to raise from the first 80, from the second 60, from the third 40 cubic ft. of water per minute; find the horse-power of the engine. *Ans.* $134\frac{31}{66}$.

216. Modulus* of a Machine.—The whole work performed by a machine consists of two parts, the *useful* work and the *lost* work. The useful work is that which the machine is designed to produce, or it is that which is employed in overcoming *useful* resistances; the lost work is that which is not wanted, but which is unavoidably produced or it is that which is spent in overcoming *wasteful* resistances. For instance in drawing a train of cars, the useful work is performed in moving the train, but the lost work is that which is done in overcoming the friction of the train, the resistance of gravity on up grades, the resistance of the air, etc. The work applied to a machine is equal to the whole work done by the machine, both useful and lost, therefore the useful work is always less than the work applied to the machine.

---

* Sometimes called Efficiency. (Art. 108.)

The Modulus of a machine is the ratio of the useful work done to the work applied. Thus, if the work applied to an engine be 40 horse-power, and the engine delivers only 30 horse-power, the modulus is $\frac{3}{4}$, *i. e.*, one-quarter of the work applied to the machine is lost by friction, etc.

Let W be the work applied to the machine, W_u the useful work, and m the modulus. Then we have from the above definition

$$m = \frac{W_u}{W}. \qquad (1)$$

If a machine were *perfect*, *i. e.*, if there were no lost work, the modulus would be unity; but in every machine, some of the work is lost in overcoming wasteful resistances, so that the modulus is always less than unity; and it is of course the object of inventors and improvers to bring this fraction as near to unity as possible.

EXAMPLES.

1. An engine, of N effective horse-power, is found to pump A cubic ft. of water per min., from a mine a fathoms deep; find the modulus of the pumps.

$$\text{Work of the engine per min.} = 33000\,N \text{ H.-P.}$$

The useful work, or work expended in pumping water,

$$= 62 \cdot 5\ \mathrm{A} \times 6a = 375\ \mathrm{A} \cdot a;$$

hence from (1) we have

$$M = \frac{375\ \mathrm{A} \cdot a}{33000\,N} = \frac{\mathrm{A} \cdot a}{88\,N}, \quad \textit{Ans.}$$

2. There were A cubic ft. of water in a mine whose depth is a fathoms, when an engine of N horse-power began to work the pump; the water continued to flow into the mine at the rate of B cubic ft. per minute; required the time

in which the mine would be cleared of water, the modulus of the pump being m.

Let $x =$ the number of minutes to clear the mine of water. Then

$$\text{weight of water to be pumped} = 62 \cdot 5\,(\mathrm{A} + \mathrm{B}x)\,;$$

$$\text{work in pumping water} = 375a\,(\mathrm{A} + \mathrm{B}x) \text{ foot-pounds}\,;$$

$$\text{effective work of the engine} = m \cdot N \cdot 33000x\,;$$

$$\therefore\quad 33000\, m\, Nx = 375a\,(\mathrm{A} + \mathrm{B}x)\,;$$

$$\therefore\quad x = \frac{\mathrm{A} \cdot a}{88\, mN - \mathrm{B} \cdot a},\ \textit{Ans.}$$

3. An engine has a 6 foot cylinder; the shaft makes 30 revolutions per minute; the average steam pressure is 25 lbs. per square inch; required the horse-power when the area of the piston is 1800 square inches, the modulus of the engine being $\frac{11}{12}$.

Work done in one minute $= 1800 \times 25 \times 6 \times 2 \times 30$ foot-pounds. We multiply by twice the length of the stroke, because the piston is driven both up and down in one revolution of the shaft.

The effective horse-power $= \frac{1800 \times 25 \times 12 \times 30}{33000} \times \frac{11}{12}$

$$= 450,\ \textit{Ans.}$$

4. The diameter of the piston of a steam engine is 60 ins.; it makes 11 strokes per minute; the length of each stroke is 8 ft.; the mean pressure per square in. is 15 lbs.; required the number of cubic ft. of water it will raise per hour from a depth of 50 fathoms, the modulus of the engine being 0·65.

The number of foot-pounds of useful work done in one hour and spent in raising water $= \pi \times 30^2 \times 8 \times 15 \times 11 \times 60 \times 0 \cdot 65$, therefore, etc.

Ans. 7763 cubic ft.

5. An engine is required to pump 1000000 gallons of water every 12 hours, from a mine 132 fathoms deep; find the horse-power if the modulus be $\frac{1}{12}$, and a gallon of water weighs 10 lbs. *Ans.* $363\frac{7}{11}$ H.-P.

6. What must be the horse-power of an engine working e hours per day, to supply n families with g gallons of water each per day, supposing the water to be raised to the mean height of h feet, and that a gallon of water weighs 10 lbs., the modulus being m. *Ans.* $\frac{ngh}{198000\,em}$ H.-P.

7. Water is to be raised from a mine at two different levels, viz., 50 and 80 fathoms, from the former 30 cubic ft., and from the latter 15 cubic ft. per minute; find the horse-power of the machinery that will be required, assuming the modulus to be 0·6. *Ans.* 51·14 H.-P.

8. The diameter of the piston of an engine is 80 ins., the mean pressure of the steam is 12 lbs. per square inch, the length of the stroke is 10 ft., the number of strokes made per minute is 11; how many cubic ft. of water will it raise per minute from a depth of 250 fathoms, its modulus being 0·6? *Ans.* 42·46 cubic ft.

9. If the engine in the last example had raised 55 cubic ft. of water per minute from a depth of 250 fathoms, what would have been its modulus? *Ans.* 0·7771.

10. How many strokes per minute must the engine in Ex. 8 make in order to raise 15 cubic ft. of water per minute from the given depth? *Ans.* 4.

11. What must be the length of the stroke of an engine whose modulus is 0·65, and whose other dimensions and conditions of working are the same as in Ex. 8, if they both do the same quantity of useful work? *Ans.* 9·23 ft.

217. Kinetic and Potential Energy. Stored Work.—*The energy of a body means its power of doing work; and the total amount of energy possessed by the body is measured by the total amount of work which it is capable of doing in passing from its present condition to some standard condition.*

Every moving body possesses energy, for it can be made to do work by parting with its velocity. The velocity of the body may be used for causing it to ascend vertically against the attraction of the earth, *i. e.*, to do work against the resistance of gravity. A cannon ball in motion can penetrate a resisting body; water flowing against a water-wheel will turn the wheel; the moving air drives the ship through the water. Wherever we find matter in motion we have a certain amount of energy.

Energy, as known to us, belongs to one or the other of two classes, to which the names *kinetic* energy* and *potential energy* are given.

Kinetic energy is energy that a body possesses in virtue of its being in motion. It is energy actually in use, energy that is constantly being spent. The energy of a bullet in motion, or of a fly-wheel revolving rapidly, or of a pile-driver just before it strikes the pile, are examples of *kinetic energy.* The work done by a force on a body free to move, exerted through a given distance, is always equal to the corresponding increase of kinetic energy [Art. 189 (3)]. If a mass, m, is moving with a velocity, v, its kinetic energy is $\frac{1}{2}mv^2$ [(3) of Art. 189]. If this velocity be generated by a constant force, P, acting through a space, s, we have, (Art. 211)

$$Ps = \tfrac{1}{2}mv^2, \qquad (1)$$

that is, the work done on the body is the exact equivalent of the kinetic energy, and the kinetic energy is reconvertible into the work; and the exact amount of work which the mass m, with a velocity v, can do against resistance before its motion is completely destroyed is $\frac{1}{2}mv^2$. This is called *stored work*,* and is the amount of work that any opposing force, P, will have to do on the body before bringing it to rest. Thus, when a heavy fly-wheel is in rapid motion, a considerable portion of the work of the engine must have gone to produce this motion; and before the engine can come to a state of rest all the work stored up in the fly-wheel, as well as in the other parts of the machine, must be destroyed. In this way a fly-wheel acts as a *reservoir* of work.

---

* Called also *actual energy*, or *energy of motion.*

If a body of mass m, moving through a space s, change its velocity from v to v_0 the work done on the body as it moves through that space, (Art. 189), is

$$\tfrac{1}{2}m\,(v^2 - v_0^2). \qquad (2)$$

If the body is not perfectly free, *i. e.*, if there is one force urging the body on, and another force resisting the body, the kinetic energy, $\frac{1}{2}mv^2$, gives the excess of the work done by the former force over that done by the latter force. Thus, when the resistance of friction is overcome, the moving forces do work in overcoming this resistance, and all the work done, in *excess* of that, is *stored* in the moving mass.

Potential energy is energy that a body possesses in virtue of its position. The energy of a bent watch-spring, which does work in uncoiling; the energy of a weight raised above the earth, as the weight of a clock which does work in falling; the energy of compressed air, as in the air-gun, or in an air-brake on a locomotive, which does work in expanding; the energy of water stored in a mill-dam, and of steam in a boiler, are all examples of *potential energy.*

---

* Called also *accumulated work*, See Todhunter's Mechs., also stored energy and not work. Browne's Mechanics, p. 178.

Such energy may or may not be called into action, it may be dormant for years; the power exists, but the action will begin only when the weight, or the water, or the steam is released. Hence the word potential, is significant, as expressing that the energy is in existence, and that a new power has been conferred upon it by the act of raising or confining it.

For example suppose a weight of 1 lb. be projected vertically upwards with a velocity of 32·2 ft. per second. The energy imparted to the body will carry it to a height of 16·1 ft., when it will cease to have any velocity. The whole of its kinetic energy will have been expended; but the body will have acquired potential energy instead; *i. e.*, the kinetic energy of the body will all have been converted into potential energy, which, if the weight be lodged for any time, is stored up and ready to be freed whenever the body shall be permitted to fall, and bring it back to its starting point with the velocity of 32·2 ft. per second; and thus the body will reacquire the kinetic energy which it originally received. Hence kinetic energy and potential energy are mutually convertible.

Let h be the height through which a body must fall to acquire the velocity v, m and W the mass and weight, respectively. Then since $v^2 = 2gh$, we have, for the stored work,

$$\tfrac{1}{2}mv^2 = \frac{W}{2g}v^2 = \frac{W}{2g}\cdot 2gh = Wh. \tag{3}$$

Hence we may say that the work stored in a moving body is measured *by the product of the weight of the body into the height through which it must fall to acquire the velocity.*

EXAMPLES.

1. Let a bullet leave the barrel of a gun with the velocity of 1000 ft. per second, and suppose it to weigh 2 ozs.; find

the work stored up in the bullet, and the height from which it must fall to acquire that velocity.

Here we have from (3) for the stored work

$$\frac{2}{2 \times 16g}(1000)^2 = Wh$$

$$= 1941 \text{ foot-pounds.}$$

$$\therefore \quad h = 15528 \text{ feet.}$$

2. A ball weighing w lbs. is projected along a horizontal plane with the velocity of v ft. per second; what space, s, will the ball move over before it comes to a state of rest, the coefficient of friction being f?

Here the resistance of friction is fw, which acts directly opposite to the motion, therefore the work done by friction while the body moves over s feet $= fws$; the work stored up in the ball $= \frac{1}{2}mv^2 = \frac{wv^2}{2g}$; therefore from (1) we have

$$fws = \frac{wv^2}{2g}; \qquad \therefore \quad s = \frac{v^2}{2gf}.$$

3. A railway train, weighing T tons, has a velocity of v ft. per second when the steam is turned off; what distance, s, will the train have moved on a level rail, whose friction is p lbs. per ton, when the velocity is v_0 ft. per second?

Here the work done by friction $= pTs$; hence from (2) we have

$$pTs = \tfrac{1}{2} \cdot \frac{2240\,T}{g}(v^2 - v_0^2);$$

$$\therefore \quad s = \frac{1120\,(v^2 - v_0^2)}{gp}.$$

4. A train of T tons descends an incline of s ft. in length, having a total rise of h ft.; what will be the velocity, v, acquired by the train, the friction being p lbs. per ton?

Here we have (Art. 213, Sch. 2), the work done on the train = the work of gravity — the work of friction

$$= 2240\,Th - pTs;$$

which is equal to the work stored up in the train. Hence

$$\frac{2240\,Tv^2}{2g} = 2240\,Th - pTs;$$

$$\therefore\quad v = \sqrt{2gh - \tfrac{1}{1120}gps}.$$

5. If the velocity of the train in the last example be v_0 ft. per second when the steam is turned off, what will be its velocity, v, when it reaches the bottom of the incline?

Ans. $v = \sqrt{{v_0}^2 + 2gh - \tfrac{1}{1120}gps}.$

6. A body weighing 40 lbs. is projected along a rough horizontal plane with a velocity of 150 ft. per sec.; the coefficient of friction is $\frac{1}{8}$; find the work done against friction in five seconds. *Ans.* 3500 foot-pounds.

7. Find the work accumulated in a body which weighs 300 lbs. and has a velocity of 64 ft. per second.

Ans. 19200 foot-pounds.

218. Kinetic Energy of a Rigid Body revolving round an Axis.—Let m be the mass of any particle of the body at the distance r from the axis, and let ω be the angular velocity, which will be the same for every particle, since the body is rigid; then the kinetic energy of $m = \frac{1}{2}m\,(r\omega)^2$. The kinetic energy of the whole body will be found by taking the sum of these expressions for every particle of the body. Hence it may be written

$$\Sigma\,\tfrac{1}{2}mr^2\omega^2 = \frac{\omega^2}{2}\,\Sigma\,mr^2. \qquad (1)$$

$\Sigma\, mr^2$ is called the *moment of inertia* of the body about the axis, and will be explained in the next chapter.

Hence *the kinetic energy of any rotating body* $= \frac{1}{2}I\omega^2$, *where* I *is the moment of inertia round the axis, and* ω *the angular velocity.*

In the case of a *fly-wheel,* it is sufficient in practice to treat the whole weight as distributed uniformly along the circumference of the circle described by the mean radius of the rim. Let r be this radius; then the moment of inertia of any particle of the wheel $= mr^2$, and the moment of inertia of the whole wheel $= Mr^2$, where M is the total mass. Hence, substituting in (1) we have $\frac{\omega^2}{2} Mr^2$, which is the kinetic energy of the fly-wheel.

EXAMPLES.

1. Two equal particles are made to revolve on a vertical axis at the distances of a and b feet from it; required the point where the two particles must be collected so that the work may not be altered.

Let $m =$ the mass of each particle, $k =$ the distance of the required point from the axis, and $\omega =$ the angular velocity; then we have

Work stored in both particles $= \frac{1}{2}m\,(a\omega)^2 + \frac{1}{2}m\,(b\omega)^2$;

Work stored in both particles collected at point $= m\,(k\omega)^2$;

$$\therefore\quad m\,(k\omega)^2 = \tfrac{1}{2}m\,(a\omega)^2 + \tfrac{1}{2}m\,(b\omega)^2;$$

$$\therefore\quad k = \sqrt{\tfrac{1}{2}\,(a^2 + b^2)}.$$

This point is called the *centre of gyration.* (See next chapter.)

2. The weight of a fly-wheel is w lbs., the wheel makes n revolutions per minute, the diameter is $2r$ feet, diameter

of axle a inches, and the coefficient of friction on the axle f; how many revolutions, x, will the wheel make before it stops?

$$\text{Work stored in the wheel} = \frac{w}{2g}\left(\frac{2\pi n}{60}\right)^2 r^2,$$

$$= \frac{w}{2g}\,\frac{\pi^2 n^2 r^2}{900}.$$

Work done by friction in x revolutions

$$= fw\frac{\pi a}{12}x;$$

and when the wheel stops, we have

$$fw\frac{\pi a}{12}x = \frac{w}{2g}\,\frac{\pi^2 n^2 r^2}{900};$$

$$\therefore\ x = \frac{\pi n^2 r^2}{150 fag}.$$

3. Required the number of strokes, x, which the fly-wheel in the last example, will give to a forge hammer whose weight is W lbs. and lift h feet, supposing the hammer to make one lift for every revolution of the wheel.

Here the work due to raising hammer $= Whx$. $\therefore$ &c.

$$\textit{Ans.}\ x = \frac{w\,\pi^2 n^2 r^2}{150g\,(12\,Wh + \pi a\,fw)}.$$

4. The weight of a fly-wheel is 8000 lbs., the diameter 20 feet, diameter of axle 14 inches, coefficient of friction 0.2; if the wheel is separated from the engine when making 27 revolutions per minute, find how many revolutions it will make before it stops (g taken $= 32.2$).

Ans. 16.9 revolutions.

219. Force of a Blow.—In order to express the amount of force between the face of a hammer, for instance, and the head of a nail, we must consider what weight must be laid upon the head of the nail to force it into the wood. A nail requires a large force to pull it out, when friction alone is retaining it, and to force it in must of course require a still larger force.

Now the head of the hammer, when it delivers a blow upon the head of the nail, must be capable of developing a force equal for a short time to the continued pressure that would be produced by a very heavy load. Hence, the effect of the hammer may be explained by the principles of *energy.* When the hammer is in motion it has a quantity of kinetic energy stored up in it, and when it comes in contact with the nail this energy is instantly converted into work which forces the nail into the wood.

EXAMPLES.

1. Suppose that a hammer weighs 1 lb., and that it is impelled downwards by the arm with considerable force, so that, at the instant the head of the hammer reaches the nail, it is moving with a velocity of 20 ft. per second ; find the force which the hammer exerts on the nail if it is driven into the wood one-tenth of an inch.

Let P be the force which the hammer exerts on the nail, then the work done in forcing the nail into the wood = $P \times \frac{1}{120}$, and the energy stored up in the hammer

$$= \tfrac{1}{2}mv^2 = \frac{(20)^2}{64} = 6.2.$$

Since the work done in forcing the nail into the wood must be equal to all the work stored in the hammer, (Art. 217), we have

$$\frac{P}{120} = 6.2\,; \quad \therefore \quad P = 744 \text{ lbs.}$$

Hence the force which the hammer exerts on the head of the nail is at least 744 lbs.

2. If the hammer in the last example forces the nail into the wood only 0.01 of an inch, find the force exerted on the nail. *Ans.* 7440 lbs.

Hence, we see that, according as the wood is harder, *i. e.*, according as the nail enters less at each stroke, the force of the blow becomes greater. So that when we speak of the "force of a blow," we must specify how soon the body giving the blow will come to rest, otherwise the term is meaningless. Thus, suppose a ball of 100 lbs. weight have a velocity that will cause it to ascend 1000 ft.; if the ball is to be stopped at the end of 1000 ft., a force of 100 lbs. will do it; but if it is to be stopped at the end of one foot, it will need a force of 100000 lbs. to do it; and to stop it at the end of one inch will require a force of 1200000 lbs., and so on.

220. Work of a Water-Fall.—When water or any body falls from a given height, it is plain that the work which is stored up in it, and which it is capable of doing, is equal to that which would be required to raise it to the height from which it has fallen; *i. e.*, if 1 lb. of water descend through 1 foot it must accumulate as much work as would be required to raise it through 1 foot. Hence when a fall of water is employed to drive a water-wheel, or any other hydraulic machine, whose modulus is given, the work done upon the machine is equal to the weight of the water in pounds × its fall in feet × the modulus of the machine.

EXAMPLES.

1. The breadth of a stream is b feet, depth a feet, mean velocity v feet per minute, and the height of the fall h feet; find (1) the horse-power, N, of the water-wheel whose modulus is m, and (2) find the number of cubic feet, A, which the wheel will pump per minute from the bottom of the fall to the height of h_1 feet.

Weight of water going over the fall per min. $= 62.5\,abv$.

$$\therefore \quad \text{Work of wheel per min.} = 62.5\,abvhm. \qquad (1)$$

$$\therefore \quad N = \frac{62.5\,abvhm}{33000}. \qquad (2)$$

Work in pumping water per min. $= 62.5\,Ah_1$;

which must = the work of the wheel per min.; hence from (1) we have

$$62.5\,Ah_1 = 62.5\,abvhm;$$

$$\therefore \quad A = \frac{abvhm}{h_1}. \qquad (3)$$

2. The mean section of a stream is 5 ft. by 2 ft.; its mean velocity is 35 ft. per minute; there is a fall of 13 ft. on this stream, at which is erected a water-wheel whose modulus is 0.65; find the horse-power of the wheel.

Ans. 5.6 H.-P.

3. In how many hours would the wheel in Ex. 2 grind 8000 bushels of wheat, supposing each horse-power to grind 1 bushel per hour? *Ans.* $1428\frac{4}{7}$ hours.

4. How many cubic feet of water must be made to descend the fall per minute in Ex. 2, that the wheel may grind at the rate of 28 bushels per hour?

Ans. 1750 cu. ft.

5. Given the stream in Ex. 2, what must be the height of the fall to grind 10 bushels per hour, if the modulus of the wheel is 0.4? *Ans.* 37.7 feet.

6. Find the useful horse-power of a water-wheel, supposing the stream to be 5 ft. broad and 2 ft. deep, and to flow with a velocity of 30 ft. per minute; the height of the fall being 14 ft., and the modulus of the machine 0.65.

Ans. 5.2 nearly.

221. The Duty of an Engine.—*The duty of an engine is the number of units of work which it is capable of doing by burning a given quantity of fuel.*—It has been found by experiment that, whatever may be the pressure at which the steam is formed, the quantity of fuel necessary to evaporate a given volume of water is always nearly the same; hence it is most advantageous to employ steam of a high pressure.*

In good ordinary engines the duty varies between 200000 and 500000 units of work for a lb. of coal. The extent to which the economy of fuel may be carried is well illustrated by the engines employed to drain the mines in Cornwall, England. In 1815, the average duty of these engines was 20 million units of work for a bushel† of coal; in 1843, by reason of successive improvements, the average duty had become 60 millions, effecting a saving of £85000 per annum. It is stated that in the case of one engine, the duty was raised to 125 millions. The duty of the engine depends largely on the construction of the boiler; 1 lb. of coal in the Cornish boiler evaporates $11\frac{1}{2}$ lbs. of water, while in a differently-shaped boiler 8.7 is the maximum.‡

EXAMPLES.

1. An engine burns 2 lbs. of coal for each horse-power per hour; find the duty of the engine for a lb. of coal.

Here the work done in one hour

$$= 60 \times 33000 \text{ foot-pounds};$$

therefore the duty of the engine $= 30 \times 33000$ foot-pounds,

$$= 990000 \text{ foot-pounds}.$$

2. How many bushels of coal must be expended in a day of 24 hours in raising 150 cubic ft. of water per minute

---

* See Tate in Mechanics' Magazine, in the year 1841.

† One bushel of coal = 84 or 94 lbs., depending upon where it is. Goodeve, p. 120.

‡ Bourne on the Steam Engine, p. 171, and Fairbairn, Useful Information, p. 177

from a depth of 100 fathoms; the duty of the engine being 60 millions for a bushel of coal?

Ans. 135 bushels.

3. A steam engine is required to raise 70 cubic ft. of water per minute from a depth of 800 ft.; find how many tons of coal will be required per day of 24 hours, supposing the duty of the engine to be 250000 for a lb. of coal.

Ans. 9 tons.

222. Work of a Variable Force.—When the force which performs work through a given space varies, the work done may be determined by multiplying the given space by the mean of all the variable forces.

Let AG represent the space in units of feet through which a variable force is exerted. Divide AG into six equal parts, and suppose P_1, P_2, P_3, etc., to be the forces in pounds applied at the points A, B, C, etc., respectively. At these points draw the ordinates y_1, y_2, y_3, etc., to represent the forces which act at the points A, B, C, etc. Then the work done from A to B will be equal to the space, AB, multiplied by the mean of the forces P_0 and P_1, *i. e.*, the work will be represented by the area of the surface A*ab*B. In like manner the work done from B to C will be represented by the area B*bc*C, and so on; so that the work done through the whole space, AG, by a force which varies continuously, will be represented by the area A*ag*G. This area may be found approximately by the ordinary rule of *Mensuration* for the area of a curved surface with equidistant ordinates, or more accurately by Simpson's* rule, the proof of which we shall now give.

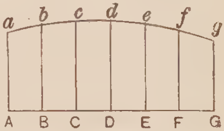

Fig. 89

223. Simpson's Rule.—Let y_1, y_2, y_3, etc., be the equidistant ordinates (Fig. 89) and l the distance between any two consecutive ordinates; then by taking the sum of the trapezoids, A*ab*B, B*bc*C, etc., we have for the area of A*ag*G,

---

* Although it was not invented by Simpson. See Todhunter.

$$\tfrac{1}{2}l\,(y_1 + y_2) + \tfrac{1}{2}l\,(y_2 + y_3) + \tfrac{1}{2}l\,(y_3 + y_4) + \text{etc.}$$

$$= \tfrac{1}{2}l\,(y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + y_7); \quad (1)$$

which is the ordinary formula of mensuration.

Now it is evident that when the curve is always concave to the line AG (1) will give too small a result, and if convex it will give too large a result.

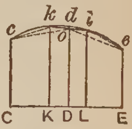

Fig. 90

Let Fig. 90 represent the space between any two odd consecutive ordinates, as C*c* and E*e* (Fig. 89); divide CE into three equal parts, CK = KL = LE, and erect the ordinates K*k* and L*l*, dividing the two trapezoids C*cd*D and D*de*E into the three trapezoids C*ck*K, K*kl*L, and L*le*E. The sum of the areas of these three trapezoids

$$= \tfrac{1}{2}\text{CK}\,(\text{C}c + 2\text{K}k + 2\text{L}l + \text{E}e)$$

$$= \tfrac{1}{3}l\,(\text{C}c + 2\text{K}k + 2\text{L}l + \text{E}e), \text{ (since } \tfrac{1}{2}\text{CK} = \tfrac{1}{3}\text{CD} = \tfrac{1}{3}l)$$

$$= \tfrac{1}{3}l\,(\text{C}c + 4\text{D}o + \text{E}e), \text{ (since } 2\text{K}k + 2\text{L}l = 4\text{D}o), \quad (2)$$

which is a closer approximation for the area of C*ce*E than (1).

Now when the curve is concave towards CE, (2) is smaller than the area between CE and the curve *ckdle*; if we substitute for D*o*, the ordinate D*d*, which is a little greater than D*o* and which is given, (2) becomes

$$\tfrac{1}{3}l\,(\text{C}c + 4\text{D}d + \text{E}e), \quad (3)$$

which is a still closer approximation than (2).

Similarly we have for the areas of A$ac$C and E$eg$G,

$$\tfrac{1}{3}l\,(\mathrm{A}a + 4\mathrm{B}b + \mathrm{C}c),\text{ and }\tfrac{1}{3}l\,(\mathrm{E}e + 4\mathrm{F}f + \mathrm{G}g). \qquad (4)$$

Adding (3) and (4) together, we have for an approximate value of the whole area,

$$\tfrac{1}{3}l\,[y_1 + y_7 + 2\,(y_3 + y_5) + 4\,(y_2 + y_4 + y_6)], \qquad (5)$$

which is Simpson's Formula. Hence Simpson's rule for finding the area approximately is the following: *Divide the abscissa,* AG, *into an even number of equal parts, and erect ordinates at the points of division; then add together the first and last ordinates, twice the sum of all the other odd ordinates, and four times the sum of all the even ordinates; multiply the sum by one-third of the common distance between any two adjacent ordinates.* (See Todhunter's Mensuration, also Tate's Geometry and Mensuration, also Morin's Mech's, by Bennett.)

EXAMPLES.

1. A variable force has acted through 3 ft.; the value of the force taken at seven successive equidistant points, including the first and the last, is in lbs. 189, 151.2, 126, 108, 94.5, 84, 75.6; find the whole work done.

Ans. 346.4 foot-pounds.

2. A variable force has acted through 6 ft.; the value of the force taken at seven successive equidistant points, including the first and the last, is in lbs. 3, 8, 15, 24, 35, 48, 63; find the whole work done.

Ans. 162 foot-pounds.

3. A variable force has acted through 9 ft.; the value of the force taken at seven successive equidistant points, including the first and the last, is in lbs. 6.082, 6.164, 6.245, 6.325, 6.403, 6.481, 6.557; find the whole work done.

Ans. 56.907 foot-pounds.

Should any of the ordinates become zero, it will not prevent the use of Simpson's rule.

Simpson's rule is applicable to other investigations as well as to that of work done by a variable force. For example, if we want the velocity generated in a given time in a particle by a variable force, let the straight line AG represent the whole time during which the force acts, and let the straight lines at right angles to AG represent the force at the corresponding instants; then the area will represent the whole space described in the given time.

EXAMPLES.

1. The ram of a pile-driving engine weighs half a ton,* and has a fall of 17 ft.; how many units of work are performed in raising this ram? *Ans.* 19040 foot-pounds.

2. How many units of work are required to raise 7 cwt. of coal from a mine whose depth is 13 fathoms?

Ans. 61152 foot-pounds.

3. A horse is used to lift the earth from a trench, which he does by means of a cord going over a pulley. He pulls up, twice every 5 minutes, a man weighing 130 lbs., and a barrowful of earth weighing 260 lbs. Each time the horse goes forward 40 ft.; find the units of work done by the horse per hour. *Ans.* 374400.

4. A railway train of T tons ascends an inclined plane which has a rise of e ft. in 100 ft., with a uniform speed of m miles per hour; find the horse-power of the engine, the friction being p lbs. per ton.

Ans. $\frac{mT(p + 22.4e)}{375}$ H.-P.

5. A railway train of 80 tons ascends an incline which rises one foot in 50 ft., with the uniform rate of 15 miles

---

* One ton = 20 cwt. = 2240 lbs.

per hour; find the horse-power of the engine, the friction being 8 lbs. per ton. *Ans.* 168.96 H.-P.

6. If a horse exert a traction of t lbs., what weight, w, will he pull up or down a hill of small inclination which has a rise of e in 100, the coefficient being f?

$$\textit{Ans.}\ w = \frac{100t}{100f \pm e}.$$

7. From what depth will an engine of 22 horse-power raise 13 tons of coal in an hour? *Ans.* 1496 ft.

8. An engine is observed to raise 7 tons of material an hour from a mine whose depth is 85 fathoms; find the horse-power of the engine, supposing $\frac{1}{6}$ of its work to be lost in transmission. *Ans.* 4.8465 H.-P.

9. Required the horse-power of an engine that would supply a city with water, working 12 hours a day, the water to be raised to a height of 50 ft.; the number of inhabitants being 130000, and each person to use 5 gallons of water a day, the gallon weighing $8\frac{1}{3}$ lbs. nearly.

Ans. 11.4 H.-P.

10. From what depth will an engine of 20 horse-power raise 600 cubic feet of water per hour? *Ans.* 1056 feet.

11. At what rate per hour will an engine of 30 horse-power draw a train weighing 90 tons gross, the resistance being 8 lbs. per ton? *Ans.* 15.625 miles.

12. What is the gross weight of a train which an engine of 25 horse-power will draw at the rate of 25 miles an hour, resistances being 8 lbs. per ton?

Ans. 46.875 tons.

13. A train whose gross weight is 80 tons travels at the rate of 20 miles an hour; if the resistance is 8 lbs. per ton, what is the horse-power of the engine?

Ans. $34\frac{2}{15}$ H.-P.

14. What must be the length of the stroke of a piston of an engine, the surface of which is 1500 square inches, which makes 20 strokes per minute, so that with a mean pressure of 12 lbs. on each square inch of the piston, the engine may be of 80 horse-power? *Ans.* $7\frac{1}{3}$ ft.

15. The diameter of the piston of an engine is 80 ins., the length of the stroke is 10 ft., it makes 11 strokes per minute, and the mean pressure of the steam on the piston is 12 lbs. per square inch; what is the horse-power?

Ans. 201·06 H.-P.

16. The cylinder of a steam engine has an internal diameter of 3 ft., the length of the stroke is 6 ft., it makes 6 strokes per minute; under what effective pressure per square inch would it have to work in order that 75 horse-power may be done on the piston? *Ans.* 67·54 lbs.

17. It is said that a horse, walking at the rate of $2\frac{1}{2}$ miles an hour, can do 1650000 units of work in an hour; what force in pounds does he continually exert?

Ans. 125 lbs.

18. Find the horse-power of an engine which is to move at the rate of 30 miles an hour, the weight of the engine and load being 50 tons, and the resistance from friction 16 lbs. per ton. *Ans.* 64 H.-P.

19. There were 6000 cubic ft. of water in a mine whose depth is 60 fathoms, when an engine of 50 horse-power began to work the pump; the engine continued to work 5 hours before the mine was cleared of the water; required the number of cubic ft. of water which had run into the mine during the 5 hours, supposing $\frac{1}{4}$ of the work of the engine to be lost by transmission. *Ans.* 10500 cubic ft.

20. Find the horse-power of a steam engine which will raise 30 cubic ft. of water per minute from a mine 440 ft. deep. *Ans.* 25 H.-P.

21. If a pit 10 ft. deep with an area of 4 square ft. be excavated and the earth thrown up, how much work will have been done, supposing a cubic foot of earth to weigh 90 lbs. *Ans.* 18000 ft.-lbs.

22. A well-shaft 300 ft. deep and 5 ft. in diameter is full of water; how many units of work must be expended in getting this water out the well; (*i. e.*, irrespectively of any other water flowing in)? *Ans.* 55223262 ft.-lbs.

23. A shaft a ft. deep is full of water; find the depth of the surface of the water when one-quarter of the work required to empty the shaft has been done. *Ans.* $\frac{a}{2}$ ft.

24. The diameter of the cylinder of an engine is 80 ins., the piston makes per minute 8 strokes of $10\frac{1}{4}$ ft. under a mean pressure of 15 lbs. per square inch; the modulus of the engine is $0 \cdot 55$; how many cubic ft. of water will it raise from a depth of 112 ft. in one minute?

Ans. $485 \cdot 78$ cub. ft.

25. If in the last example the engine raised a weight of 66433 lbs. through 90 ft. in one minute, what must be the mean pressure per square inch on the piston?

Ans. $26 \cdot 37$ lbs.

26. If the diameter of the piston of the engine in Ex. 24 had been 85 ins., what addition in horse-power would that make to the *useful* power of the engine?

Ans. $13 \cdot 28$ H.-P.

27. If an engine of 50 horse-power raise 2860 cub. ft. of water per hour from a mine 60 fathoms deep, find the modulus of the engine. *Ans.* $\cdot 65$.

28. Find at what rate an engine of 30 horse-power could draw a train weighing 50 tons up an incline of 1 in 280, the resistance from friction being 7 lbs. per ton.

Ans. 1320 ft. per minute.

29. A forge hammer weighing 300 lbs. makes 100 lifts a minute, the perpendicular height of each lift being 2 ft.; what is the horse-power of the engine that gives motion to 20 such hammers? *Ans.* 36·36 H.-P.

30. An engine of 10 horse-power raises 4000 lbs. of coal from a pit 1200 ft. deep in an hour, and also gives motion to a hammer which makes 50 lifts in a minute, each lift having a perpendicular height of 4 ft.; what is the weight of the hammer? *Ans.* 1250 lbs.

31. Find the horse-power of the engine to raise T tons of coal per hour from a pit whose depth is a fathoms, and at the same time to give motion to a forge hammer weighing w lbs., which makes n lifts per minute of h ft. each.

Ans. $\dfrac{224aT + nhw}{33000}$ H.-P.

32. Find the useful work done by a fire engine per second which discharges every second 13 lbs. of water with a velocity of 50 ft. per second. *Ans.* 508 nearly.

33. A railway truck weighs m tons; a horse draws it along horizontally, the resistance being n lbs. per ton; in passing over a space s the velocity changes from u to v; find the work done by the horse in this space.

Ans. $\dfrac{2240m}{2g}(v^2 - u^2) + mns$.

34. The weight of a ram is 600 lbs., and at the end of the blow has a velocity of $32\frac{1}{6}$ ft.; what work has been done in raising it? *Ans.* 9650.

35. Required the work stored in a cannon ball whose weight is $32\frac{1}{6}$ lbs., and velocity 1500 ft. *Ans.* 1125000.

36. A ball, weighing 20 lbs., is projected with a velocity of 60 ft. a second, on a bowling-green; what space will the ball move over before it comes to rest, allowing the friction to be $\frac{1}{18}$ the weight of the ball? *Ans.* 1007·3 ft.

37. A train, weighing 193 tons, has a velocity of 30 miles an hour when the steam is turned off; how far will the train move on a level rail before coming to rest, the friction being $5\frac{1}{2}$ lbs. per ton? *Ans.* 12256 ft.

38. A train, weighing 60 tons, has a velocity of 40 miles an hour, when the steam is turned off, how far will it ascend an incline of 1 in 100, taking friction at 8 lbs. a ton? *Ans.* $3942\frac{1}{2}$ ft.

39. A carriage of 1 ton moves on a level rail with the speed of 8 ft. a second; through what space must the carriage move to have a velocity of 2 ft., supposing friction to be 6 lbs. a ton? *Ans.* 348 ft.

40. If the carriage in the last example moved over 400 feet before it comes to a state of rest, what is the resistance of friction per ton? *Ans.* 5.57 lbs.

41. A force, P, acts upon a body parallel to the plane; find the space, s, moved over when the body has attained a given velocity, v, the coefficient of friction being f, and the body weighing w lbs. *Ans.* $s = \frac{wv^2}{2g(P - fw)}$.

42. Suppose the body in the last example to be moved for t seconds; required (1) the velocity, v, acquired, and (2) the work stored.

Ans. (1) $\frac{P - fw}{w} tg$; (2) $\frac{(P - fw)^2}{2w} t^2 g$.

43. A body, weighing 40 lbs., is projected along a rough horizontal plane with a velocity of 150 ft. per second; the coefficient of friction is $\frac{1}{8}$; find the work done against friction in 5 seconds. *Ans.* 3500 foot-pounds.

44. A body weighing 50 lbs., is projected along a rough horizontal plane with the velocity of 40 yards per second; find the work expended when the body comes to rest.

Ans. 11250 ft.-lbs.

45. If a train of cars weighing 100000 lbs. is moving on a horizontal track with a velocity of 40 miles an hour when the steam is turned off; through what space will it move before it is brought to rest by friction, the friction being 8 lbs. per ton? *Ans.* 13374.8 ft.

46. What amount of energy is acquired by a body weighing 30 lbs. that falls through the whole length of a rough inclined plane, the height of which is 30 ft., and the base 100 ft., the coefficient of friction being $\frac{1}{5}$? *Ans.* 300 ft.-lbs.

47. If a train of cars, weighing T tons, ascend an incline having a raise of e ft. in 100 ft., with the velocity v_0 ft. per second when the steam is turned off; through what space, x, will it move before it comes to a state of rest, the friction being p lbs. per ton? *Ans.* $x = \dfrac{1120 v_0^2}{g\,(22.4e + p)}$.

48. Suppose the train, in Ex. 4, Art. 217, to be attached to a rope, passing round a wheel at the top of the incline, which has an empty train of T_1 tons attached to the other extremity of the rope; what velocity, v, will the train acquire in descending s ft. of the incline?

$$Ans.\ v = \sqrt{2gh\,\frac{T - T_1}{T + T_1} - \frac{gps}{1120}}.$$

49. An engine of 35 horse-power makes 20 revolutions per minute, the weight of the fly-wheel is 20 tons and the diameter is 20 ft.; what is the accumulated energy in foot-pounds? *Ans.* 307054.

50. If the fly-wheel in the last example lifted a weight of 4000 lbs. through 3 ft., what part of its angular velocity would it lose? *Ans.* $\frac{1}{51}$.

51. If the axis of the above fly-wheel be 6 ins. in diameter, the coefficient of friction 0·075, what fraction,

approximately, of the 35 horse-power is expended in turning the fly-wheel? *Ans.* $\frac{1}{11}$.

52. In pile driving, 38 men raised a ram 12 times in an hour; the weight of the ram was 12 cwt., and the height through which it was raised 140 ft.; find the work done by one man in a minute. *Ans.* 990 ft.-lbs.

53. A battering-ram, weighing 2000 lbs., strikes the head of a pile with a velocity of 20 ft. per second; how far will it drive the pile if the constant resistance is 10000 lbs.? *Ans.* 1.25 ft.

54. A nail 2 ins. long was driven into a block by successive blows from a monkey weighing 5.01 lbs.; after one blow it was found that the head of the nail projected 0·8 of an inch above the surface of the block; the monkey was then raised to a height of 1.5 ft., and allowed to fall upon the head of the nail; after this blow the head of the nail was 0.46 of an inch above the surface; find the force which the monkey exerted upon the head of the nail at this blow. *Ans.* 265.24 lbs.

55. The monkey of a pile-driver, weighing 500 lbs. is raised to a height of 20 ft., and then allowed to fall upon the head of a pile, which is driven into the ground 1 inch by the blow; find the force which the monkey exerted upon the head of the pile. *Ans.* 120000 lbs.

56. A steam hammer, weighing 500 lbs., falls through a height of 4 ft. under the action of its own weight and a steam pressure of 1000 lbs.; find the amount of work which it can do at the end of the fall. *Ans.* 6000 ft.-lbs.

57. The mean section of a stream is 8 square ft.; its mean velocity is 40 ft. per minute; it has a fall of 17½ ft.; it is required to raise water to a height of 300 ft. by means of a water-wheel whose modulus is 0.7; how many cubic ft. will it raise per minute? *Ans.* 13.07 cub. ft.

58. To what height would the wheel in the last example raise $2\frac{1}{4}$ cub. ft. of water per minute? *Ans.* $1742\frac{2}{9}$ ft.

59. The mean section of a stream is $1\frac{1}{2}$ ft. by 11 ft.; its mean velocity is $2\frac{1}{2}$ miles an hour; there is on it a fall of 6 ft. on which is erected a wheel whose modulus is 0.7; this wheel is employed to raise the hammers of a forge, each of which weighs 2 tons, and has a lift of $1\frac{1}{2}$ ft.; how many lifts of a hammer will the wheel yield per minute?

Ans. 142 nearly.

60. In the last example determine the mean depth of the stream if the wheel yields 135 lifts per minute.

Ans. 1.43 ft.

61. In Ex. 59, how many cubic ft. of water must descend the fall per minute to yield 97 lifts of the hammer per minute? *Ans.* 2483 cub. ft.

62. A stream is a ft. broad and b ft. deep, and flows at the rate of c ft. per hour; there is a fall of h ft.; the water turns a machine of which the modulus is e; find the number of bushels of corn which the machine can grind in an hour, supposing that it requires m units of work per minute for one hour to grind a bushel. *Ans.* $\dfrac{1000abche}{16 \times 60m}$.

63. Down a 14-ft. fall 200 cub. ft. of water descend every minute, and turn a wheel whose modulus is 0.6. The wheel lifts water from the bottom of the fall to a height of 54 ft.; (1) how many cubic ft. will be thus raised per minute? (2) If the water were raised from the top of the fall to the same point, what would the number of cubic ft. then be? *Ans.* (1) 31.1 cub. ft.; (2) 34.7 cub. ft.

In the second case the number of cub. ft. of water taken from the top of the fall being x, the number of ft. that will turn the wheel will be $200 - x$.

64. Find how many units of work are stored up in a

mill-pond which is 100 ft. long, 50 ft. broad, and 3 ft. deep, and has a fall of 8 ft. *Ans.* 7500000.

65. There are three distinct levels to be pumped in a mine, the first 100 fathoms deep, the second 120, the third 150; 30 cub. ft. of water are to come from the first, 40 from the second, and 60 from the third per minute; the duty of the engine is 70000000 for a bushel of coal. Determine (1) its working horse-power and (2) the consumption of coal per hour. *Ans.* (1) 191 H.-P.; (2) 5.4 bushels.

66. In the last example suppose there is another level of 160 fathoms to be pumped, that the engine does as much work as before for the other levels, and that the utmost power of the engine is 275 H.-P.; find the greatest number of cub. ft. of water that can be raised from the fourth level. *Ans.* $46\frac{1}{4}$ cub. ft.

67. A variable force has acted through 8 ft.; the value of the force taken at nine successive equidistant points, including the first and the last, is in lbs. 10.204, 9.804, 9.434, 9.090, 8.771, 8.475, 8.197, 7.937, 7.692; find the whole work done. *Ans.* 70.641 foot-pounds.

68. The value of a variable force, taken at nine successive equidistant points, including the first and the last points, is in lbs. 2.4849, 2.5649, 2.6391, 2.7081, 2.7726, 2.8332, 2.8904, 2.9444, 2.9957, the common distance between the points is 1 ft.; find the whole work done. *Ans.* 22.0957 foot-pounds.

69. A train whose weight is 100 tons (including the engine) is drawn by an engine of 150 horse-power, the friction being 14 lbs. per ton, and all other resistances neglected; find the maximum speed which the engine is capable of sustaining on a level rail. *Ans.* $40\frac{5}{28}$ miles per hour.

70. If the train described in the last example be moving at a particular instant with a velocity of 15 miles per hour,

and the engine working at full power, what is the acceleration at that instant? (Call $g = 32$.) *Ans.* $\frac{47}{140}$.

71. Find the horse-power of an engine required to drag a train of 100 tons up an incline of 1 in 50 with a velocity of 30 miles an hour, the friction being 1400 lbs.

Ans. The engine must be of not less than $470\frac{2}{5}$ horse-power. This is somewhat above the power of most locomotive engines.

72. A train, of 200 tons weight, is ascending an incline of 1 in 100 at the rate of 30 miles per hour, the friction being 8 lbs. per ton. The steam being shut off and the break applied, the train is stopped in a quarter of a mile. Find the weight of the break-van, the coefficient of friction of iron on iron being $\frac{1}{6}$. *Ans.* $11\frac{3}{14}$ tons.

CHAPTER VI.

MOMENT OF INERTIA.*

224. Moments of Inertia.—The quantity Σmr^2 in which m is the mass of an element of a body, and r its distance from an axis, occurs frequently in problems of rotation, so that it becomes necessary to consider it in detail; it is called *the moment of inertia* of the body about the axis (Art. 218). Hence, "moment of inertia" may be defined as follows: *If the mass of every particle of a body be multiplied by the square of its distance from a straight line, the sum of the products so formed is called the Moment of Inertia of the body about that line.*

If the mass of every particle of a body be multiplied by the square of its distance from a given plane or from a given point, the sum of the products so formed is called the moment of inertia of the body with reference to that plane or that point.

If the body be referred to the axes of x and y, and if the mass of each particle be multiplied by its *two* co-ordinates, x, y, the sum of the products so formed is called the *product of inertia* of the body about those two axes.

If dm denote the mass of an element, p its distance from the axis, and I the moment of inertia, we have

$$I = \Sigma p^2 dm. \qquad (1)$$

If the body be referred to rectangular axes, and x, y, z, be the co-ordinates of any element, then, according to the definitions, the moments of inertia about the axes of x, y, z, respectively, will be

---

* This term was introduced by Euler, and has now got into general use whenever Rigid Dynamics is studied.

$$\Sigma\,(y^2 + z^2)\,dm, \quad \Sigma\,(z^2 + x^2)\,dm, \quad \Sigma\,(x^2 + y^2)\,dm. \qquad (2)$$

The moments of inertia with respect to the planes yz, zx, xy respectively, are,

$$\Sigma\,x^2dm, \quad \Sigma\,y^2dm, \quad \Sigma\,z^2dm. \qquad (3)$$

The products of inertia with respect to the axes y and z, z and x, x and y, are

$$\Sigma\,yzdm, \quad \Sigma\,zxdm, \quad \Sigma\,xydm. \qquad (4)$$

The moment of inertia with respect to the origin is

$$\Sigma\,(x^2 + y^2 + z^2)\,dm = \Sigma\,r^2dm, \qquad (5)$$

where r is the distance of the particle from the origin.

The moment of inertia of a lamina, when the axis lies in it, is called a *rectangular moment of inertia*, and when it is perpendicular to the lamina it is called a *polar moment of inertia*, and the corresponding axis is called the *rectangular* or the *polar* axis.

The process of finding *moments and products of inertia* is merely that of integration; but after this has been accomplished for the simplest axes possible, they can be found without integration for any other axes.

EXAMPLES.

1. Find the moment of inertia of a uniform rod, of mass m, and length l, about an axis through its centre at right angles to it.

Let x be the distance of any element of the rod from the centre, and μ the mass of a unit of length; then $dm = \mu dx$, which in (1) gives for the moment of inertia $\Sigma\,\mu x^2dx$, or

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} \mu x^2dx,$$

remembering that the symbol of summation, Σ, includes integration in the cases wherein the body is a continuous mass.

Hence $$I = \tfrac{1}{12}\mu l^3 = \tfrac{1}{12}ml^2.$$

If the axis be drawn through one end of the rod and perpendicular to its length we shall have for the moment of inertia

$$I = \tfrac{1}{3}ml^2.$$

2. Find the moment of inertia of a rectangular lamina* about an axis through its centre, parallel to one of its sides.

Let b and d denote the breadth and depth respectively of the rectangle, the former being parallel to the axis. Imagine the lamina composed of elementary strips of length b parallel to the axis. Let the distance of one of them from the axis be y, and its breadth dy; then, denoting the mass of a unit of area by μ, we have $dm = \mu b dy$, which in (1) gives

$$I = \int_{-\frac{1}{2}d}^{\frac{1}{2}d} \mu b y^2 dy = \tfrac{1}{12}\mu b d^3 = \tfrac{1}{12}md^2.$$

If the axis be drawn through one end of the rectangle, we shall have for the moment of inertia

$$I = \tfrac{1}{3}md^2.$$

3. Find the moment of inertia of a circular lamina with respect to an axis through its centre and perpendicular to its surface.

Let the radius $= a$, and μ the mass of a unit of area as before, then we have

$$I = \int_0^{2\pi}\int_0^a \mu r^3\, dr\, d\theta = \frac{\pi\mu a^4}{2} = \frac{ma^2}{2}.$$

---

* In all cases we shall assume the thickness of the laminæ or plates to be infinitesimal.

4. Find the moment of inertia of a circular plate (1) about a diameter as an axis, and (2) about a tangent.

Ans. (1) $\frac{1}{4}ma^2$; (2) $\frac{5}{4}ma^2$.

5. Find the moment of inertia of a square plate, (1) about an axis through its centre and perpendicular to its plane, (2) about an axis which joins the middle points of two opposite sides, and (3) about an axis passing through an angular point of the plate, and perpendicular to its plane. Let a = the side of the plate and μ the mass of a unit of area.

$$(1)\ I = \int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}} \mu\,(x^2 + y^2)\,dy\,dx = \frac{\mu a^4}{6} = \frac{m}{6}\,a^2;$$

(2) $\frac{1}{12}ma^2$; (3) $\frac{2}{3}ma^2$.

6. Find the moment of inertia of an isosceles triangular plate, (1) about an axis through its vertex and perpendicular to its plane, and (2) about an axis which passes through its vertex and bisects the base.

Let $2b$ = the base and a = the altitude, then

$$I = 2\int_0^a\int_0^{\frac{b}{a}x} \mu\,(x^2 + y^2)\,dy\,dx = \frac{m}{6}\,(3a^2 + b^2);\ (2)\ \tfrac{1}{6}mb^2.$$

225. Moments of Inertia relative to Parallel Axes, or Planes.—*The moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through the centre of gravity of the body, plus the product of the mass of the body into the square of the distance between the axes.*

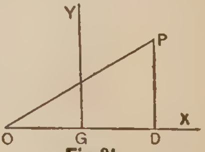

Fig. 91

Let the plane of the paper pass through the centre of gravity of the body, and be perpendicular to the two parallel axes, meeting them in O and G, and let P be the projection of any element on the plane of the paper.

Take the centre of gravity, G, as origin, the fixed axis through it perpendicular to the plane of the paper as the axis of z, and the plane through this and the parallel axis for that of zx; and let I_1 be the moment of inertia about the axis through G, I that for the parallel axis through O, a the distance, OG, between the axes, and (x, y) any point, P. Then we shall have

$$I_1 = \Sigma\,(x^2 + y^2)\,dm\,;\ I = \Sigma\,[(x + a)^2 + y^2]\,dm.$$

Hence $$I - I_1 = 2a\,\Sigma x dm + a^2\,\Sigma dm = a^2 m,$$

since $\Sigma x dm = 0$, as the centre of gravity is at the origin.

$$\therefore\quad I = I_1 + a^2 m, \tag{1}$$

which is called *the formula of reduction.*

Hence the moment of inertia of a body relative to any axis can be found when that for the parallel axis through its centre of gravity is known.

Cor. 1.—The moments of inertia of a body are the same for all parallel axes situated at the same distance from its centre of gravity. Also, of all parallel axes, that which passes through the centre of gravity of a body has the least moment of inertia.

Cor. 2.—It is evident that the same theorem holds if the moments of inertia be taken with respect to parallel planes, instead of parallel axes.

A similar property also connects the moment of inertia relative to any point with that relative to the centre of gravity of the body.

EXAMPLES.

1. The moment of inertia of a rectangle* in reference to an axis through its centre and parallel to one end is

---

* See Note to Ex. 2, Art. 224; strictly speaking, an *area* has a moment of inertia no more than it has weight.

$\frac{1}{12}md^2$; find the moment of inertia in reference to a parallel axis through one end.

From (1) we have

$$I = \tfrac{1}{12}md^2 + \frac{d^2}{4}m = \tfrac{1}{3}md^2.$$

2. The moment of inertia of an isosceles triangle about an axis through its vertex and perpendicular to its plane is $\frac{1}{6}m(3a^2 + b^2)$, (Art. 224, Ex. 6); find its moment about a parallel axis through the centre.

From (1) we have

$$I_1 = \tfrac{1}{6}m(3a^2 + b^2) - \tfrac{4}{9}a^2m = \tfrac{1}{6}m(\tfrac{1}{3}a^2 + b^2).$$

3. Find the moment of inertia of a circle about an axis through its circumference and perpendicular to its plane (See Ex. 3, Art. 224). *Ans.* $\frac{3}{2}ma^2$.

4. Find the moment of inertia of a square about an axis through the middle point of one of its sides and perpendicular to its plane (Ex. 5, Art. 224). *Ans.* $\frac{5}{12}ma^2$.

226. Radius of Gyration.—Let k be such a quantity that the moment of inertia $= mk^2$, then we shall have

$$I = \Sigma r^2 dm = mk^2. \qquad (1)$$

The distance k is called the *radius of gyration* of the body with respect to the fixed axis, and it denotes the distance from the axis to that point into which if the whole mass were concentrated the moment of inertia would not be altered. The *point* into which the body might be concentrated, without altering its moment of inertia, is called the *centre of gyration*. When the fixed axis passes through the centre of gravity, the *length* k and the *point* of concentration are called *principal radius* and *principal centre of gyration*.

Let k_1 = the principal radius of gyration and r_1 the distance of an element from the axis through the centre of gravity; then from (1) we have

$$mk^2 = \Sigma\, r^2 dm$$

$$= \Sigma\, r_1{}^2 dm + ma^2, \text{ [by (1) of Art. 225]}$$

$$= mk_1{}^2 + ma^2;$$

$$\therefore\quad k^2 = k_1{}^2 + a^2, \tag{2}$$

from which it appears that the *principal radius of gyration is the least radius for parallel axes,* which is also evident from Cor. 1, Art. 225.

SCH.—In homogeneous bodies, since the mass of any part varies directly as its volume, (1) may be written

$$\Sigma\, r^2 dV = Vk^2, \tag{3}$$

where dV denotes the element of volume, and V the entire volume of the body.

Hence, in homogeneous bodies, the value of k is independent of the density of the body, and depends only on its form; and in determining the moment of inertia, we may take the element of volume or weight for the element of mass, and the total volume or weight of the body instead of its mass.

Also in finding the moment of inertia of a lamina, since k is independent of the thickness of the lamina, we may take the element of area instead of the element of mass, and the total area of the lamina instead of its mass.

From (1) we have

$$k^2 = \frac{I}{m}. \tag{4}$$

Similarly,

$$k_1{}^2 = \frac{I_1}{m}, \tag{5}$$

hence, *the square of the radius of gyration with respect to any axis equals the moment of inertia with respect to the same axis divided by the mass.*

EXAMPLES.

1. Find the principal radius of gyration of a straight line.

From Ex. 1, Art. 224, we have

$$I_1 = \tfrac{1}{12}ml^2;$$

therefore from (5) we have $k_1{}^2 = \frac{1}{12}l^2$.

2. Find the principal radius of gyration of a circle (1) with respect to a polar axis, and (2) with respect to a rectangular axis. *Ans.* (1) $\frac{1}{2}a^2$; (2) $\frac{1}{4}a^2$.

3. Find the principal radius of gyration of a rectangle with respect to a rectangular axis. *Ans.* $\frac{1}{12}d^2$.

4. Find the principal radius of gyration (1) of a square with respect to a polar axis, and (2) of an isosceles triangle with respect to a polar axis.

Ans. (1) $\frac{1}{6}a^2$; (2) $\frac{1}{6}(\frac{1}{3}a^2 + b^2)$.

227. Polar Moment of Inertia.—If any thin plate, or lamina, be referred to two rectangular axes and x, y be the co-ordinates of any element, then (Art. 224) the moments of inertia about the axes of x and y respectively, are $\Sigma\, y^2 dm$ and $\Sigma\, x^2 dm$; and therefore the moment of inertia with respect to the axis drawn perpendicular to the plane at the intersection of the axes of x and y is

$$\Sigma\,(x^2 + y^2)\,dm.$$

Hence *the polar moment of inertia of any lamina is equal to the sum of the moments of inertia with respect to any two rectangular axes, lying in the plane of the lamina.*

Cor.—For every two rectangular axes in the plane of the lamina, at any point, we have

$$\Sigma x^2dm + \Sigma y^2dm = \text{const.}$$

that is, *the sum of the moments of inertia with respect to a pair of rectangular axes is constant.* Hence, if one be a maximum, the other is a minimum, and *vice versa.*

EXAMPLES.

1. Find the moment of inertia of a rectangle with respect to an axis through its centre and perpendicular to its plane.

From Ex. 2, Art. 224, the rectangular moments of inertia are

$$\tfrac{1}{12}md^2 \text{ and } \tfrac{1}{12}mb^2;$$

therefore the polar moment of inertia $= \frac{1}{12}m(d^2 + b^2)$; $k_1{}^2 = \frac{1}{12}(d^2 + b^2)$.

2. Find the moment of inertia of an isosceles triangle with respect to an axis through its centre parallel to its base, a being the altitude and and $2b$ the base.

Ans. $\frac{1}{18}ma^2$; $k_1{}^2 = \frac{1}{18}a^2$.

228. Moment of Inertia of a Solid of Revolution, with respect to its Geometric Axis.—Let the axis be that of x; and let the equation of the generating curve be $y = f(x)$. Let the solid be divided into an infinite number of circular plates perpendicular to the axis of revolution; let the density be uniform and μ the mass of a unit of volume; and denote by x the distance of the centre of any circular plate from the origin, y its radius, and dx its thickness; then the moment of inertia of this circular plate about an axis through its centre and perpendicular to its plane, by (Ex. 3, Art. 224), is

$$\frac{\pi\mu y^4 dx}{2} = \frac{\pi\mu}{2}[f(x)]^4 dx;$$

therefore the moment of inertia of the whole solid is

$$\frac{\pi\mu}{2}\int [f(x)]^4 dx; \qquad (1)$$

the integration being taken between proper limits.

EXAMPLES.

1. Find the moment of inertia of a right circular cone about its axis.

Let h = the height and b = the radius of the base; then the equation of the generating curve is $y = \frac{b}{h}x$, which in (1) gives for the moment of inertia,

$$I = \frac{\pi\mu b^4}{2h^4}\int_0^h x^4 dx = \frac{\pi\mu h b^4}{10}$$

$$= \tfrac{3}{10} m b^2, \left(\text{since } m = \frac{\pi}{3}\mu h b^2\right).$$

Therefore $k_1{}^2 = \frac{3}{10}b^2$.

2. Find the moment of inertia (1) of a solid cylinder about its axis, b being its radius and h its height, and (2) of a hollow cylinder, b and b' being the external and internal radii. *Ans.* (1) $\frac{1}{2}mb^2$; (2) $\frac{1}{2}m(b^2 + b'^2)$.

3. Find the moment of inertia of a paraboloid about its axis, h being its altitude and b the radius of the base.

Ans. $\frac{\pi\mu h b^4}{6}$.

229. Moment of Inertia of a Solid of Revolution, with respect to an Axis Perpendicular to its Geometric Axis.—Take the origin at the intersection of the axis of revolution with the axis about which the moment of inertia is required; and denoting by x the distance of the centre of any circular plate from the origin, y its radius and dx its thickness, we have for the moment of inertia of this circular plate, about a diameter, by Ex. 4, Art. 224,

$$\frac{\pi\mu y^4}{4}\,dx;$$

therefore (Art. 225) the moment of inertia of this plate about the parallel axis at the distance x from it is

$$\frac{\pi\mu y^4}{4}\,dx + \pi\mu\, y^2x^2\,dx;$$

therefore the moment of inertia of the whole solid is

$$\pi\mu\int\left(\frac{y^4}{4} + y^2x^2\right)dx, \qquad (1)$$

the integration being taken between proper limits.

EXAMPLES.

1. Find the moment of inertia of a right circular cone about an axis through its vertex and perpendicular to its own axis.

Let h = the height and b = the radius of the base, then the moment of inertia from (1)

$$= \pi\mu\int_0^h\left(\frac{b^4}{4h^4} + \frac{b^2}{h^2}\right)x^4dx = \frac{\pi\mu h b^2}{20}\,(4h^2 + b^2)$$

$$= \tfrac{3}{20}m\,(4h^2 + b^2).$$

2. Find the moment of inertia of a cone, whose altitude $= h$, and the radius of whose base $= b$, about an axis through its centre of gravity and perpendicular to its own axis. *Ans.* $\tfrac{3}{80}m\,(h^2 + 4b^2)$.

3. Find the moment of inertia of a paraboloid of revolution about an axis through its vertex and perpendicular to its own axis, the altitude being h and the radius of the base b.

Ans. $\frac{\pi\mu h b^2}{12}(b^2 + 3h^2)$.

230. Moment of Inertia of Various Solid Bodies.

EXAMPLES.

1. Find the moment of inertia of a rectangular parallelopiped about an axis through its centre of gravity and parallel to an edge.

Let the edges be a, b, c; since a parallelopiped may be conceived as consisting of an infinite number of rectangular laminæ, each of which has the same radius of gyration relative to an axis perpendicular to its plane, it follows that the radius of gyration of the parallelopiped is the same as that of the laminæ. Hence, the moments of inertia relative to three axes through the centre and parallel to the edges a, b, c, respectively, are by Ex. 1, Art. 227, $\frac{1}{12}m(b^2 + c^2)$, $\frac{1}{12}m(a^2 + c^2)$, $\frac{1}{12}m(a^2 + b^2)$.

2. Find the moment of inertia of a rectangular parallelopiped about an edge.

This may be obtained immediately from the last example by using Art. 225, or otherwise independently as follows:

Take the three edges a, b, c for the axes of x, y, z, respectively; let μ be the mass of a unit of volume, then the moment of inertia relative to the edge a is

$$= \int_0^a \int_0^b \int_0^c \mu (y^2 + z^2)\, dx\, dy\, dz$$

$$= \frac{\mu abc}{3}(b^2 + c^2) = \tfrac{1}{3}m(b^2 + c^2);$$

and similarly for the moments of inertia about the edges b and c.

The moment of inertia of a cube whose edge is a with respect to one of its edges is $\frac{2}{3}\mu a^5 = \frac{2}{3}ma^2$.

3. Find the moment of inertia of a segment of a sphere relative to a diameter parallel to the plane of section, the radius of the sphere being a and the distance of the plane section from the centre b.

Ans. $\frac{1}{60}\pi\mu\,(16a^5 + 15a^4b + 10a^2b^3 - 9b^5)$.

231. Moment of Inertia of a Lamina with respect to any Axis.—When the moment of inertia of a plane figure about any axis is known, we easily find the moment of inertia about any parallel axis (Art. 225); also, when the moments of inertia about two rectangular axes in the plane of the figure are known, the moment of inertia about the straight line at right angles to the plane of these axes at their intersection is known immediately, (Art. 227); we now proceed to find the moment of inertia about any straight line in the plane inclined to these axes at *any* angle.

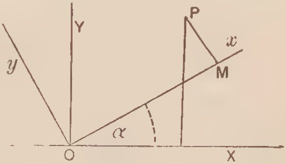

Fig. 92

Through any point, O, as origin, draw two rectangular axes, OX, OY, in the plane of the lamina; and draw any straight line, Ox, in the plane. It is required to find the moment of inertia about Ox in terms of the moments of inertia about OX and OY.

Let P be any point of the lamina, x, y, its rectangular, and r, θ, its polar co-ordinates, $p = PM$, and α the angle xOX. Then if I be the moment of inertia of the lamina relative to Ox, a and b the moments of inertia relative to the axes of x and y respectively, and h the product of inertia relative to the same axes, we have

$$\begin{aligned} I &= \Sigma\, p^2 dm = \Sigma\, r^2 \sin^2 (\theta - \alpha)\, dm \\ &= \Sigma\, (y \cos \alpha - x \sin \alpha)^2\, dm \\ &= \cos^2 \alpha\, \Sigma\, y^2 dm + \sin^2 \alpha\, \Sigma\, x^2 dm - 2 \sin \alpha \cos \alpha\, \Sigma\, xydm \\ &= a \cos^2 \alpha + b \sin^2 \alpha - 2h \sin \alpha \cos \alpha. \qquad (1) \end{aligned}$$

If we choose the axes so that the term h or $\Sigma\, xydm = 0$, the expression for I becomes much simpler. The pair of axes so chosen are called the *principal axes* at the point; and the corresponding moments of inertia are called the *principal moments of inertia* of the lamina, relative to the point.

If A and B represent these principal moments of inertia, (1) becomes

$$I = A \cos^2 \alpha + B \sin^2 \alpha. \qquad (2)$$

Hence, *the moment of inertia of a lamina with respect to any axis through a point may be found when the principal moments with respect to the point are determined.*

232. Principal Axes of a Body.—*At any point of a rigid body and in any plane there is a pair of principal axes.*

Let OX, OY (Fig. 92), be any rectangular axes in the plane; let Ox, Oy, be another set of rectangular axes in the same plane, inclined to the former at an angle α; let a, b, and h, as before, denote the moments and product of inertia about OX, OY, and let (x', y') be any point, P, referred to the axes Ox, Oy. Then, using the notation of the last article, we have

$$x' = r \cos (\theta - \alpha); \quad y' = r \sin (\theta - \alpha);$$

$$\begin{aligned} \Sigma\, x'y'dm &= \tfrac{1}{2} \Sigma\, r^2 \sin 2 (\theta - \alpha)\, dm \\ &= \cos 2\alpha\, \Sigma\, r^2 \sin \theta \cos \theta\, dm \\ &\quad - \tfrac{1}{2} \sin 2\alpha\, \Sigma\, r^2 (\cos^2 \theta - \sin^2 \theta)\, dm. \end{aligned}$$

Putting this $= 0$, and solving for α, we obtain

$$\tan 2\alpha = \frac{2\Sigma\, r^2 \sin\theta \cos\theta\, dm}{\Sigma\, r^2 (\cos^2\theta - \sin^2\theta)\, dm}$$

$$= \frac{2\Sigma\, xy\, dm}{\Sigma\, (x^2 - y^2)\, dm} = \frac{2h}{b - a} \qquad (1)$$

As the tangent of an angle may have any value, positive or negative, from 0 to ∞, it follows that (1) will always give a real value for 2α, so there is always a set of principal axes; that is, *at every point in a body there exists one pair of rectangular axes for which the quantity h or $\Sigma\, xy\, dm = 0$.*

Cor.—It may also be shown that at every point of a rigid body there are *three* axes at right angles to one another, for which the products of inertia vanish.*

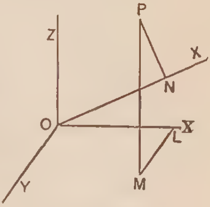

Fig. 91*a*

* Let a, b, c, be the moments of inertia about three axes, OX, OY, OZ, at right angles to one another; d, e, f, the products of inertia ($\Sigma\, myz$, $\Sigma\, mzx$, $\Sigma\, mxy$, respectively). Let Ox be any line drawn through the origin, making angles α, β, γ, with the co-ordinate axes.

Let OL, LM, MP, be the co-ordinates x, y, z, of any point P of the body at which an element of mass m is situated. Draw PN perpendicular to Ox.

Projecting the broken line, OLMP, on ON, (Art. 102), we have

$$\text{ON} = x\cos\alpha + y\cos\beta + z\cos\gamma;$$

also $$\text{OP}^2 = x^2 + y^2 + z^2, \quad \text{and} \quad 1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma.$$

The moment of inertia I about $Ox = \Sigma\, m\text{PN}^2$

$$= \Sigma m\, (\text{OP}^2 - \text{ON}^2)$$

$$= \Sigma m\, [x^2 + y^2 + z^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2]$$

$$= \Sigma m\, [(x^2+y^2+z^2)(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)-(x\cos\alpha+y\cos\beta+z\cos\gamma)^2]$$

$$= \Sigma m\, (y^2 + z^2)\cos^2\alpha + \Sigma m\, (z^2 + x^2)\cos^2\beta + \Sigma m\, (x^2 + y^2)\cos^2\gamma$$

$$- 2\Sigma myz\cos\beta\cos\gamma - 2\Sigma mzx\cos\gamma\cos\alpha - 2\Sigma m\cos\alpha\cos\beta$$

$$= a\cos^2\alpha + b\cos^2\beta + c\cos^2\gamma - 2d\cos\beta\cos\gamma$$

$$- 2e\cos\gamma\cos\alpha - 2f\cos\alpha\cos\beta. \qquad (1)$$

To represent this geometrically, take a point Q on ON; and let its distance from O be r, and its co-ordinates be x_1, y_1, z_1. Then

$$x_1 = r\cos\alpha, \quad y_1 = r\cos\beta, \quad z_1 = r\cos\gamma.$$

SCH.—In many cases the position of the principal axes can be seen at once. Suppose, for example, we wish the principal axis for a rectangle when the given point is the centre. Draw through the centre straight lines parallel to the sides of the rectangle; then these will be the principal

---

Therefore (1) becomes

$$I = \frac{ax_1^2 + by_1^2 + cz_1^2 - 2dy_1z_1 - 2ez_1x_1 - 2fx_1y_1}{r^2}. \tag{2}$$

But the equation

$$ax_1^2 + by_1^2 + cz_1^2 - 2dy_1z_1 - 2ez_1x_1 - 2fx_1y_1 = 1, \tag{3}$$

denotes an ellipsoid whose centre is at O; because a, b, c are necessarily positive, since a moment of inertia is essentially positive, being the sum of a number of squares. If then Q is a point on this ellipsoid, (2) becomes

$$I = \Sigma m PN^2 = \frac{1}{r^2};$$

or the moment of inertia about any line through O, is measured by the square of the reciprocal of the radius vector of this ellipsoid, which coincides with the line.

This is called the *momental ellipsoid*, and was first used by Cauchy, *Exercises de Math., Vol. II.* It has no physical existence, but is an artifice to bring under the methods of geometry the properties of moments of inertia. The momental ellipsoid has a definite form for every point of a rigid body.

Now every ellipsoid has three axes, to which if it is referred, the coefficients of yz, zx, xy vanish, and therefore (3), when transformed to these axes takes the form

$$Ax_1^2 + By_1^2 + Cz_1^2 = 1; \tag{4}$$

and hence (1) or (2) when referred to these axes, becomes

$$I = A\cos^2 \alpha + B\cos^2 \beta + C\cos^2 \gamma, \tag{5}$$

where A, B, C, are the moments of inertia of the body about these axes.

When three rectangular axes, meeting in a given point, are chosen so that the products of inertia all vanish, they are called the *principal axes* at the given point.

The three planes through any two principal axes are called the *principal planes* at the given point.

The moments of inertia about the principal axes at any point are called the *principal moments of inertia* at that point.

If the three principal moments of inertia of a body are equal to one another, the ellipsoid (4) becomes a sphere, since A = B = C; and therefore the moment of inertia about every other axis is equal to these, for (5) becomes

$$I = A(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = A;$$

and every axis is a principal axis. (See Routh's Rigid Dynamics, p. 12, Price's Anal. Mech's, Vol. II, p. 156, Pirie's Rigid Dynamics, p. 76, etc.)

axes; because for every element, dm, on one side of the axis of x at the point (x, y), there is another element of equal mass on the other side at the point $(x, -y)$. Hence, $\Sigma\, xy\, dm$ consists of terms which may be arranged in pairs, so that the two terms in a pair are numerically equal but of opposite signs; and therefore $\Sigma\, xy\, dm = 0$.

Again, if in any uniform body a straight line can be drawn with respect to which the body is exactly symmetrical, this must be a principal axis at every point in its length. Any diameter of a uniform circle or sphere or the axis of a parabola or ellipse or hyperbola is a principal axis at any point in its line; but the diagonal of a rectangular plate is not for this reason a principal axis at its middle point, for every straight line drawn perpendicular to it is not equally divided by it.

Let the body be symmetrical about the plane of xy, then for every element dm, on one side of the plane at the point (x, y, z), there is another element of equal mass on the other side at the point $(x, y, -z)$. Hence, for such a body $\Sigma\, xz\, dm = 0$ and $\Sigma\, yz\, dm = 0$. If the body be a lamina in the plane of xy, then z of every element is zero, and we have again $\Sigma\, xz\, dm = 0$, $\Sigma\, yz\, dm = 0$.

Thus, in the case of the ellipsoid, the three principal sections are all planes of symmetry, and therefore the three axes of the ellipsoid are principal axes. Also, at every point in a lamina one principal axis is the perpendicular to the plane of the lamina.

EXAMPLES.

1. Find the moment of inertia of a rectangular lamina about a diagonal.

From Ex. 2, Art. 224, the moments of inertia about two lines through the centre parallel to the sides (principal moments of inertia) are

$$\tfrac{1}{12}md^2 \quad \text{and} \quad \tfrac{1}{12}mb^2;$$

where b and d are the breadth and depth respectively.

Also, if α be the angle which the diagonal makes with the side b, we have

$$\sin^2\alpha = \frac{d^2}{b^2+d^2}, \quad \cos^2\alpha = \frac{b^2}{b^2+d^2}.$$

Substituting these values for A, B, $\sin^2\alpha$, $\cos^2\alpha$, in (2) of Art. 231, we have

$$\begin{aligned} I &= \tfrac{1}{12}md^2\frac{b^2}{b^2+d^2} + \tfrac{1}{12}mb^2\frac{d^2}{b^2+d^2} \\ &= \tfrac{1}{6}m\,\frac{b^2d^2}{b^2+d^2}. \end{aligned}$$

2. Find the moment of inertia of an isosceles triangular plate about an axis through its centre and inclined at an angle α to its axis of symmetry, a being its altitude and $2b$ its base. *Ans.* $\tfrac{1}{6}m\,(\tfrac{1}{3}a^2\cos^2\alpha + b^2\sin^2\alpha)$.

3. Find the moment of inertia of a square plate about a diagonal, a being a side of the square. *Ans.* $\tfrac{1}{12}ma^2$.

233. Products of Inertia.—The value of the product of inertia at any point may be made to depend on the value of the product of inertia for parallel axes through the centre of gravity. Let (x, y) be the position of any element, dm, referred to axes through any assigned point; (x', y') the position of the element referred to parallel axes through the centre of gravity, and (h, k) the centre of gravity referred to the first pair of axes. Then

$$x = x' + h, \quad y = y' + k;$$

therefore

$$\begin{aligned} \Sigma\, xy\, dm &= \Sigma\,(x'+h)(y'+k)\,dm \\ &= \Sigma\, x'y'\,dm + hk\,\Sigma\, dm, \qquad (1) \end{aligned}$$

since $\Sigma\, mx' = 0$, and $\Sigma\, my' = 0$.

SCH.—By (1) we may often find the product of inertia for an assigned origin and axes. Thus, suppose we require the product of inertia in the case of a rectangle, when the origin is at the corner, and the axes are the edges which meet at that corner. By Art. 232, Sch. we have $\Sigma x'y'dm = 0$; therefore from (1) we have

$$\Sigma xydm = hk\Sigma dm;$$

and as h and k are known, being half the lengths of the edges of the rectangle to which they are respectively parallel, the product of inertia is known.

EXAMPLES.

Find the expressions for the moments of inertia in the following, the bodies being supposed homogeneous in all cases.

1. The moment of inertia of a rod of length a, with respect to an axis perpendicular to the rod and at a distance d from its middle point. *Ans.* $m\left(\frac{a^2}{12} + d^2\right)$.

2. The moment of inertia of an arc of a circle whose radius is a and which subtends an angle 2α at the centre, (1) about an axis through its centre perpendicular to its plane, (2) about an axis through its middle point perpendicular to its plane, (3) about the diameter which bisects the arc.

Ans. (1) ma^2; (2) $2m\left(1-\frac{\sin\alpha}{\alpha}\right)a^2$; (3) $m\left(1-\frac{\sin 2\alpha}{2\alpha}\right)\frac{a^2}{2}$.

3. The moment of inertia of the arc of a complete cycloid whose length is a with respect to its base. *Ans.* $\frac{1}{30}ma^2$.

4. The moment of inertia of an equilateral triangle, of side a, relative to a line in its plane, parallel to a side, at the distance d from its centre of gravity. *Ans.* $m\left(\frac{a^2}{24} + d^2\right)$.

5. Given a triangle whose sides are a, b, c, and whose perpendiculars on these sides, from the opposite vertices are p, q, r, respectively; find the moment of inertia of the triangle about a line drawn through each vertex and parallel respectively, (1) to the side a, (2) to the side b, (3) to the side c. *Ans.* (1) $\frac{1}{2}mp^2$; (2) $\frac{1}{2}mq^2$; (3) $\frac{1}{2}mr^2$.

6. Find the moment of inertia of the triangle in the last example relative to the three lines drawn through the centre of gravity of the triangle and parallel respectively to the sides a, b, c. *Ans.* $\frac{1}{18}mp^2$; $\frac{1}{18}mq^2$; $\frac{1}{18}mr^2$.

7. Find the moment of inertia of the triangle in Ex. 5, relative to the three sides a, b, c, respectively.

Ans. $\frac{1}{6}mp^2$; $\frac{1}{6}mq^2$; $\frac{1}{6}mr^2$.

8. The moment of inertia of a right angled triangle, of hypothenuse c, relative to a perpendicular to its plane passing through the right angle. *Ans.* $\frac{1}{6}mc^2$.

9. The moment of inertia of a ring whose outer and inner radii are a and b respectively, (1) with respect to a polar axis through its centre, and (2) with respect to a diameter. *Ans.* (1) $\frac{1}{2}m(a^2 + b^2)$; (2) $\frac{1}{4}m(a^2 + b^2)$.

10. The moment of inertia of an ellipse, (1) with respect to its major axis, (2) with respect to its minor axis, and (3) with respect to an axis through its centre and perpendicular to its plane.

Ans. (1) $\frac{1}{4}mb^2$; (2) $\frac{1}{4}ma^2$; (3) $\frac{1}{4}m(a^2 + b^2)$.

11. The moment of inertia of the surface of a sphere of radius a about its diameter. *Ans.* $\frac{2}{3}ma^2$.

12. The moment of inertia of a right prism whose base is a right angled triangle, with respect to an axis passing through the centres of gravity of the ends, the sides containing the right angle of the triangular base being a and b and the height of the prism c. *Ans.* $\frac{1}{18}m(a^2 + b^2)$.

13. The moment of inertia of a right prism whose height is c, about an axis passing through the centres of gravity of the ends, the base of the prism being an isosceles triangle whose base is a and height b.

Ans. $\frac{1}{6}\left(\frac{a^2}{4} + \frac{b^2}{3}\right) m$.

14. The moment of inertia of a sphere of radius a, (1) relative to a diameter, and (2) relative to a tangent.

Ans. (1) $\frac{2}{5}ma^2$; (2) $\frac{7}{5}ma^2$.

15. The moment of inertia, about its axis of rotation, (1) of a prolate spheroid, and (2) of an oblate spheroid.

Ans. (1) $\frac{2}{5}mb^2$; (2) $\frac{2}{5}ma^2$.

16. The moment of inertia of a cylinder, relative to an axis perpendicular to its own axis and intersecting it, (1) at a distance c from its end, (2) at the end of the axis, and (3) at the middle point of the axis, the altitude of the cylinder being h and radius of its base a.

Ans. (1) $\frac{1}{4}ma^2 + \frac{1}{3}m(h^2 - 3hc + 3c^2)$;
(2) $\frac{1}{12}m(3a^2 + 4h^2)$; (3) $\frac{1}{12}m(h^2 + 3a^2)$.

17. The moment of inertia of an ellipse about a central radius vector r, making an angle α with the major-axis.

Ans. $\frac{1}{4}m\frac{a^2b^2}{r^2}$.

18. The moment of inertia of the area of a parabola cut off by any ordinate at a distance x, from the vertex, (1) about the tangent at the vertex, and (2) about the axis of the parabola.

Ans. $\frac{3}{7}mx^2$; (2) $\frac{1}{5}my^2$ where y is the ordinate corresponding to x.

19. The moment of inertia of the area of the lemniscate, $\rho^2 = a^2 \cos 2\theta$, about a line through the origin in its plane and perpendicular to its axis. *Ans.* $\frac{1}{48}m(3\pi + 8)a^2$.

20. The moment of inertia of the ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

about the axis a, b, c, respectively.

Ans. (1) $\frac{1}{5}m\,(b^2 + c^2)$; (2) $\frac{1}{5}m\,(c^2 + a^2)$;
(3) $\frac{1}{5}m\,(a^2 + b^2)$.

CHAPTER VII.

ROTATORY MOTION.

234. Impressed and Effective Forces.—All forces acting on a body other than the mutual actions of the particles, are called the *Impressed Forces* that act on the body.

Thus, when a ball is thrown in vacuo, the impressed force is gravity; if a ball is rotating about a vertical axis, the impressed forces are gravity and the reaction of the axis.

The impressed or external forces are the cause of the motion and of all the other forces. Which are the impressed forces depends upon the particular system which is under consideration. The same force may be external to one system and internal to another. Thus, the pressure between the foot of a man and the deck of a ship on which he is, is external to the ship and also to the man and is the cause of his own forward motion and of a slight backward motion of the ship; but if the man and ship are considered as parts of one system the pressure is internal.

When a particle is moving as part of a rigid body, it is acted on by the external impressed forces and also by the molecular reactions of the other particles. Now if this particle were considered as separated from the rest of the body, and all the forces removed, there is some one force which, singly, would move it in the same way as before. This force is called the *Effective Force* on the particle; it is evidently the resultant of the impressed and molecular forces on the particle.

Thus, the effective force is that part of the impressed force which is effective in causing actual motion. It is the force which is required for producing the deviation from the straight line and the change of velocity. If a particle is revolving with constant velocity round a fixed axis, the effective force is the centripetal force (Art. 198). If a heavy body falls without rotation, the whole force of gravity is effective; but if it is rotating about a horizontal axis the weight goes partly to balance the pressure on the axis.

If we suppose the particle of mass m to be at the point (x, y, z) at any time, t, and resolve the forces acting on it into the three axial components, X, Y, Z, the motion may be found [Art. 168 (2)] by solving the simultaneous equations

$$m\frac{d^2x}{dt^2} = X; \quad m\frac{d^2y}{dt^2} = Y; \quad m\frac{d^2z}{dt^2} = Z. \qquad (1)$$

If we regard a rigid body as one in which the particles retain invariable positions with respect to one another, so that no external force can alter them (Art. 43), we might write down the equations of the several particles in accordance with (1), if all the forces were known. Such, however, is not the case. We know nothing of the mutual actions of the particles, and consequently cannot determine the motion of the body by calculating the motion of its particles separately. When there are several rigid bodies which mutually act and react on one another the problem becomes still more complicated.

235. D'Alembert's Principle.*—By D'Alembert's Principle, however, all the necessary equations may be obtained without writing down the equations of motion of the several particles, and without any assumption as to the nature of the mutual actions except the following, which may be regarded as a natural consequence of the laws of motion.

The internal actions and reactions of any system of rigid bodies in motion are in equilibrium among themselves.

---

* Introduced by D'Alembert in 1742.

The axial accelerations of the particle of mass m, which is moving as part of a rigid body, are $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$. Let f be their resultant, then the effective force is measured by mf. Let F and R be the resultants of the impressed and molecular forces, respectively, on the particle. Then mf is the resultant of F and R. Hence if mf be reversed, the three forces, F, R, and mf, are in equilibrium.

The same reasoning may be applied to every particle of each body of the system, thus furnishing three groups of forces, similar, respectively, to F, R, and mf; and these three groups will form a system of forces in equilibrium. Now by D'Alembert's principle the group R will itself form a system of forces in equilibrium. Whence it follows that the group F will be in equilibrium with the group mf. Hence,

If forces equal and exactly opposite to the effective forces were applied at each particle of the system, they would be in equilibrium with the impressed forces.

That is, *D'Alembert's principle asserts that the whole effective forces of a system are together equivalent to the impressed forces.*

Sch.—By this principle the solution of a problem in Kinetics is reduced to a problem in Statics as follows: We first choose the co-ordinates by means of which the position of the system in space may be fixed. We then express the effective forces on each element in terms of its co-ordinates. These effective forces, reversed, will be in equilibrium with the given impressed forces. Lastly, the equations of motion for each body may be formed, as is usually done in Statics, by resolving in three directions and taking moments about three straight lines. (See Routh's Rigid Dynamics, Pirie's Rigid Dynamics, Pratt's Mech's, Price's Anal. Mech's, Vol. II.)

236. Rotation of a Rigid Body about a Fixed Axis under the Action of any Forces.—Let any plane passing through the axis of rotation and fixed in space be taken as a plane of reference. Let m be the mass of any element of the body, r its distance from the axis, and θ the angle which a plane through the axis and the element makes with the plane of reference.

Then the velocity of m in a direction perpendicular to the plane containing the element and the axis is $r\frac{d\theta}{dt}$. The *moment of the momentum** of this particle about the axis is $mr^2\frac{d\theta}{dt}$. Hence the moment of the momenta of all the particles is

$$\Sigma\, mr^2\frac{d\theta}{dt}. \tag{1}$$

Since the particles of the body are rigidly connected, it is clear that $\frac{d\theta}{dt}$ is the same for every particle, and is the angular velocity of the body. Hence *the moment of the momenta of all the particles of the body about the axis is the moment of inertia of the body about the axis multiplied by the angular velocity.*

The acceleration of m perpendicular to the direction in which r is measured is $r\frac{d^2\theta}{dt^2}$, and therefore the moment of the moving forces of m about the axis is $mr^2\frac{d^2\theta}{dt^2}$. Hence, *the moment of the effective forces of all the particles of the body about the axis is*

$$\Sigma\, mr^2\frac{d^2\theta}{dt^2}, \tag{2}$$

which is the moment of inertia of the body about the axis multiplied by the angular acceleration.

---

* Called also *Angular Momentum*. (See Pirie's Rigid Dynamics, p. 44.)

(1) Let the forces be impulsive (Art. 202); let ω, ω', be the angular velocities just before and just after the action of the forces, and N the moment of the impressed forces about the axis of rotation, by which the motion is produced.

Then, since by D'Alembert's principle the effective forces when reversed are in equilibrium with the impressed forces, we have from (1)

$$\omega' \Sigma mr^2 - \omega \Sigma mr^2 = N;$$

$$\therefore \quad \omega' - \omega = \frac{N}{\Sigma mr^2}$$

$$= \frac{\text{moment of impulse about axis}}{\text{moment of inertia about axis}}; \tag{3}$$

that is, *the change in the angular velocity of a body, produced by an impulse, is equal to the moment of the impulse divided by the moment of inertia of the body.*

(2) Let the forces be finite. Then taking moments about the axis as before, we have from (2)

$$\frac{d^2\theta}{dt^2} = \frac{N}{\Sigma mr^2}$$

$$= \frac{\text{moment of forces about axis}}{\text{moment of inertia about axis}}; \tag{4}$$

that is, *the angular acceleration of a body, produced by a force, is equal to the moment of the force divided by the moment of inertia of the body.*

By integrating (4) we shall know the angle through which the body has revolved in a given time. Two arbitrary constants will appear in the integrations, whose values are to be determined from the given initial values of θ and $\frac{d\theta}{dt}$. Thus the whole motion can be found, and we shall consequently be able to determine the position of the body at any instant.

SCH.—It appears from (3) and (4) that the motion of a rigid body round a fixed axis, under the action of any forces, depends on (1) the moment of the forces about that axis, and (2) the moment of inertia of the body about the axis. If the whole mass of the body were concentrated into its centre of gyration (Art. 226), and attached to the fixed axis of rotation by a rod without mass, whose length is the radius of gyration, and if this system were acted on by forces having the same moment as before, and were set in motion with the same initial values of θ and the angular velocity, then the whole subsequent angular motion of the rod would be the same as that of the body. Hence, we may say briefly, that a body turning about a *fixed* axis is kinetically given when its mass and radius of gyration are known.

EXAMPLE.

A rough circular horizontal board is capable of revolving freely round a vertical axis through its centre. A man walks on and round at the edge of the board; when he has completed the circuit what will be his position in space?

Let a be the radius of the board, M and M' the masses of the board and man respectively; θ and θ' the angles described by the board and man, and F the action between the feet of the man and the board.

The equation of motion of the board by (4) is

$$Fa = Mk_1^2 \frac{d^2\theta}{dt^2}.$$

Since the action between the man and the board is continually tangent to the path described by the man, the equation of motion of the man is, by (5) of Art. 20,

$$F = M'a\frac{d^2\theta'}{dt^2}.$$

Eliminating F and integrating twice, the constant being zero in both cases, because the man and board start from rest, we get

$$Mk_1^2\theta = M'a^2\theta'. \qquad (1)$$

When the man has completed the circuit we have $\theta + \theta' = 2\pi$; also $k_1^2 = \frac{a^2}{2}$. Substituting these in (1) we get

$$\theta' = \frac{2\pi M}{2M' + M},$$

which gives the angle in space described by the man.

If $M = M'$, this becomes

$$\theta' = \tfrac{2}{3}\pi;$$

and
$$\theta = \tfrac{4}{3}\pi,$$

which is the angle in space described by the board. (See Routh's Rigid Dynamics, p. 67.)

237. The Compound Pendulum.—*A body moves about a fixed horizontal axis acted on by gravity only, to determine the motion.*

Let ABO be a section of the body made by the plane of the paper passing through G, the centre of gravity, and cutting the axis of rotation perpendicularly at O. Let $\theta =$ the angle which OG makes with the vertical OY; and let $h =$ OG, $k_1 =$ the principal radius of gyration, and $M =$ the mass of the body. Then by (4) of Art. 236, we have

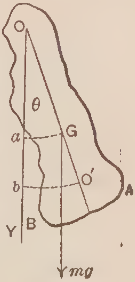

Fig. 93

$$\frac{d^2\theta}{dt^2} = \frac{Mgh \sin\theta}{\Sigma\, mr^2} = -\frac{Mgh \sin\theta}{Mk^2}$$

$$= -\frac{gh}{k_1^2 + h^2} \sin\theta \text{ [by (2) of Art. 226]}, \quad (1)$$

the negative sign being taken because θ is a decreasing function of the time.

This equation cannot be integrated in finite terms, but if the oscillations be small, we may develop $\sin\theta$ and reject all powers above the first, and (1) will become

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{k_1^2 + h^2}\theta. \quad (2)$$

Multiplying by $2\,d\theta$ and integrating, and supposing that the body began to move when θ was equal to α, (2) becomes

$$\frac{d\theta^2}{dt^2} = \frac{gh}{k_1^2 + h^2}(\alpha^2 - \theta^2).$$

Hence denoting the time of a complete oscillation by T, we have

$$T = \pi\sqrt{\frac{h^2 + k_1^2}{gh}}, \quad (3)$$

which gives the time in seconds, when h and k_1 are measured in feet and $g = 32.18$.

When a heavy body vibrates about a horizontal axis, by the force of gravity, it is called a *compound pendulum.*

Cor. 1.—If we suppose the whole mass of the compound pendulum to be concentrated into a single point, and this point connected with the axis by a medium without weight, it becomes a *simple pendulum* (Art. 194). Denoting the distance of the point of concentration from the axis by l, we have for the time of an oscillation, by (1) of Art. 194,

$\pi\sqrt{\frac{l}{g}}$. If the point be so chosen that the simple pendulum will perform an oscillation in the same time as the compound pendulum, these two expressions for the time of an oscillation must be equal to each other, and we shall have

$$l = \frac{h^2 + k_1^2}{h}$$

$$= h + \frac{k_1^2}{h} = \mathrm{OO}', \tag{4}$$

(O′ being the point of concentration).

Cor. 2.—This length is called the *length of the simple equivalent pendulum;* the point O is called the *centre of suspension*; the point O′, into which the mass of the compound pendulum must be concentrated so that it will oscillate in the same time as before, is called the *centre of oscillation;* and a line through the centre of oscillation and parallel to the axis of suspension is called *an axis of oscillation.*

From (4) we have

$$(l - h)\, h = k_1^2;$$

or

$$\mathrm{GO}' \cdot \mathrm{GO} = k_1^2. \tag{5}$$

Now (5) would not be altered if the place of O and O′ were interchanged; hence if O′ be made the centre of suspension, then O will be the centre of oscillation. Thus *the centres of oscillation and of suspension are convertible, and the time of oscillation about each is the same.*

Cor. 3.—Putting the derivative of l with respect to h in (4) equal to zero, and solving for h, we get

$$h = k_1,$$

which makes l a minimum, and therefore makes t a minimum. Hence, *when the axis of suspension passes through the principal centre of gyration the time of oscillation is a minimum.*

Rem.—The problem of determining the law under which a heavy body swings about a horizontal axis is one of the most important in the history of science. A simple pendulum is a thing of theory; our accurate knowledge of the acceleration of gravity depends therefore on our understanding the rigid or compound pendulum. This was the first problem to which D'Alembert applied his principle.

The problem was called in the days of D'Alembert, the "centre of oscillation." It was required to find if there were a point at which the whole mass of the body might be concentrated, so as to form a simple pendulum whose law of oscillation was the same.

The position of the centre of oscillation of a body was first correctly determined by Huyghens and published at Paris in 1673. As D'Alembert's principle was not known at that time, Huyghens had to discover some principle for himself.*

EXAMPLES.

1. A material straight line oscillates about an axis perpendicular to its length; find the length of the equivalent simple pendulum.

Let $2a =$ the length of the line, and h the distance of its centre of gravity from the point of suspension. Then since $k_1^2 = \frac{a^2}{3}$, we have from (4)

$$l = h + \frac{a^2}{3h}. \tag{1}$$

Cor. 1.—If the point of suspension be at the extremity of the line (1) becomes

$$l = \tfrac{4}{3}a;$$

---

* Routh's Rigid Dynamics, p. 69.

that is, the length of the equivalent simple pendulum is two-thirds of the length of the rod.

COR. 2.—Let $h = \frac{1}{3}a$; then (1) becomes

$$l = \tfrac{4}{3}a.$$

Hence, the time of an oscillation is the same, whether the line be suspended from one extremity, or from a point one-third of its length from the extremity. This also illustrates the convertibility of the centres of oscillation and of suspension (See Cor. 2).

COR. 3.—If $h = 10a$, then (1) becomes

$$l = \tfrac{301}{30}a.$$

2. A circular arc oscillates about an axis through its middle point perpendicular to the plane of the arc. Prove that the length of the simple equivalent pendulum is independent of the length of the arc, and is equal to twice the radius.

From Ex. 2, Art. 233, we have

$$k^2 = h^2 + k_1{}^2 = 2\left(1 - \frac{\sin\alpha}{\alpha}\right)a^2.$$

From Ex. 1, Art. 78, we have

$$h = a - a\frac{\sin\alpha}{\alpha}.$$

Therefore (4) becomes

$$l = 2a^2\left(1 - \frac{\sin\alpha}{\alpha}\right) \div a\left(1 - \frac{\sin\alpha}{\alpha}\right) = 2a.$$

3. A right cone oscillates about an axis passing through its vertex and perpendicular to its own axis; it is required to find the length of the simple equivalent pendulum, (1) when h is the altitude of the cone and b the radius of the base, and (2) when the altitude = the radius of the base = h.

Ans. (1) $\frac{4h^2 + b^2}{5h}$; (2) h.

That is, in the second cone, the centre of oscillation is in the centre of the base; so that the times of oscillation are equal for axes through the vertex and the centre of the base perpendicular to the axis of the cone.

4. A sphere, radius a, oscillates about an axis; find the length of the simple equivalent pendulum, (1) when the axis is tangent to the sphere, (2) when it is distant $10a$ from the centre of the sphere, and (3) when it is distant $\frac{a}{5}$ from the centre of the sphere.

Ans. (1) $\frac{7}{5}a$; (2) $\frac{251}{25}a$; (3) $\frac{11}{5}a$.

238. The Length of the Second's Pendulum Determined Experimentally.—The time of oscillation of a compound pendulum depends on $h + \frac{k_1^2}{h}$ by (4) of Art. 237. But there are difficulties in the way of determining h and k_1. The centre, G, can not be got at, and, as every body is more or less irregular and variable in density, k_1 cannot be calculated with sufficient accuracy. These quantities must therefore be determined from experiments. Bessel observed the times of oscillation about different axes, the distances between which were very accurately known. Captain Kater employed the property of the convertibility of the centres of suspension and oscillation (Art. 237, Cor. 2), as follows:

Let the pendulum consist of an ordinary straight bar, CO, and a small weight, m, which may be clamped to it by means of a screw, and shifted from one position to another on the pendulum. At the

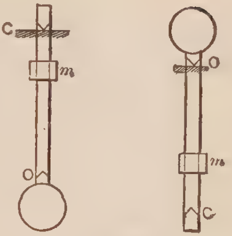

Fig. 94

points C and O in two triangular apertures, at the distance l apart, let two knife edges of hard steel be placed parallel to each other, and at right angles to the pendulum, so that it may vibrate on either of them, as in Fig. 94. Let m be shifted till it is found that the times of oscillation about C and O are exactly the same. It remains only to measure CO, and observe the time of oscillation. The distance between the two points C and O is the length of the simple equivalent pendulum. This distance between the knife edges was measured by Captain Kater with the greatest care. The mean of three measurements differed by less than a ten-thousandth of an inch from each of the separate measurements.

The time of a single vibration cannot be observed directly, because this would require the fraction of a second of time as shown by the clock, to be estimated either by the eye or ear. The difficulty may be overcome by observing the time, say of a thousand vibrations, and thus the error of the time of a single vibration is divided by a thousand. The labor of so much counting may however be avoided by the use of "the method of coincidences." The pendulum is placed in front of a clock pendulum whose time of vibration is slightly different. Certain marks made on the two pendulums are observed by a telescope at the lowest point of their arcs of vibration. The field of view is limited by a diaphragm to a narrow aperture across which the marks are seen to pass. At each succeeding vibration one pendulum follows the other more closely, and at last its mark is completely covered by the other during their passage across the field of view of the telescope. After a few vibrations it appears again preceding the other. In the interval from one disappearance to the next, one pendulum has made, as nearly as possible, one *complete* oscillation more than the other. In this manner 530 half-vibrations of a clock pendulum, each equal to a second, were found to correspond to 532 of Captain Kater's pendulum. The ratio of the times of vibration of the pendulum and the clock pendulum may thus be calculated with extreme accuracy. The rate of going of the clock must then be found by astronomical means.

The time of vibration thus found will require several corrections which are called "reductions." For instance, if the oscillation be not so small that we can put $\sin\theta = \theta$ in Art. 237, we must make a reduction to infinitely small arcs. Another reduction is necessary if we wish to reduce the result to what it would have been at the level of the sea. The attraction of the intervening land may be allowed for by Dr. Young's rule, (Phil. Trans., 1819). We may thus obtain the force of gravity at the level of the sea, supposing all the land above this level were cut off and the sea constrained to keep its present level. As the level of the sea is altered by the attraction of the land, further corrections are still necessary if we wish to reduce the result to the surface of that spheroid which most nearly represents the earth. See Routh's Rigid Dynamics, p. 77. For the details of this experiment the student is referred to the Phil. Trans. for 1818, and to Vol. X.

239. Motion of a Body when Unconstrained.—If an impulse be communicated to any point of a free body in a direction not passing through the centre of gravity, it will produce both translation and rotation.

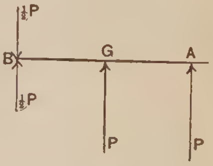

Fig.95

Let P be the impulse imparted to the body at A. At B, on the opposite side of the centre G, a distance GB = AG, let two opposite impulses be applied, each equal to $\frac{1}{2}P$; they will not alter the effect. Now if $\frac{1}{2}P$ applied at A is combined with the $\frac{1}{2}P$ at B which acts in the same direction, their resultant is P, acting at G and in the same direction, and this produces translation only. The remaining $\frac{1}{2}P$ at A combined with the remaining $\frac{1}{2}P$ at B, which acts in the opposite direction, form a couple which produces rotation about the centre G.

Hence, *when a body receives an impulse in a direction which does not pass through the centre of gravity, that centre will assume a motion of translation as though the impulse were applied immediately to it; and the body will have a motion of rotation about the centre of gravity, as though that point were fixed.*

240. Centre of Percussion.—Axis of Spontaneous Rotation.—Let Mv represent the impulse impressed upon

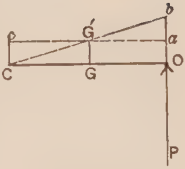

Fig. 96

the body (Fig. 96) whose mass is M, and h the perpendicular distance, GO, from the centre of gravity, G, to the line of action, OP, of the impulse. The centre of gravity will assume a motion of translation with the velocity v, in a direction parallel to that of the impulsive force. Then from (3) of Art. 236, we have for the angular velocity

$$\omega = \frac{Mvh}{Mk_1^2} = \frac{vh}{k_1^2}.$$

The absolute velocity of each point of the body will be compounded of the two velocities of translation and rotation. The point O, for example, to which the impulse is applied, has a velocity of translation, Oa, equal to that of the centre of gravity, and a velocity of rotation, ab, about the centre of gravity; so that the velocity of any point at a distance a from the centre, G, will be expressed by $v \pm a\omega$; the upper or lower sign being taken according as the point is, or is not, on the same side of the centre of gravity as the point O. Thus, if we consider the motion of the body for a very short interval of time, the line OGC will assume the position $bG'C$, the point C remaining at rest during this interval; that is, while the point C would be carried forward over the line Cc by the motion of translation, it would be carried backward through the same distance by the motion of rotation. Hence, since the absolute velocity of C is zero, we have

$$v - a\omega = 0;$$

$$\therefore \quad a = \frac{v}{\omega} = \frac{k_1^2}{h}; \tag{1}$$

and hence denoting OC by l we have

$$l = h + \frac{k_1^2}{h}. \qquad (2)$$

Now if there had been a fixed axis through C perpendicular to the plane of motion, the initial motion would have been precisely the same, and this fixed axis evidently would not have received any pressure from the impulse.

When a rigid body rotates about a fixed axis, and the body can be so struck that there is no pressure on the axis, any point in the line of action of the force is called a *centre of percussion.*

When the line of action of the blow is given and the body is free from all constraint, so that it is capable of translation as well as of rotation, the axis about which the body begins to turn is called the axis of *spontaneous rotation.* It obviously coincides with the position of the fixed axis in the first case.

Cor. 1.—From (1) we have

$$ah = GC \cdot GO = k_1^2;$$

hence the points O and C are convertible, that is, *if the axis of rotation be supposed to pass through the point* O, *the centre of spontaneous rotation will coincide with the centre of percussion.*

Cor. 2.—From (2) it follows, by comparison with (4) of Art. 237, that *if the axis of spontaneous rotation coincides with the axis of suspension, the centre of percussion coincides with the centre of oscillation.*

Sch.—It is evident that if there be a fixed obstacle at O, and it be struck by the body OC rotating about a fixed axis through C, the obstacle will receive the whole force of the moving body, and the axis will not receive any. Hence the *centre of percussion* also determines the position in which a fixed obstacle must be placed, on which if the rotating body impinges and is brought to rest, the axis of rotation will suffer no pressure.

An axis through the centre of gravity, parallel to the axis of spontaneous rotation, is called the *axis of instantaneous rotation.* A free body rotates about this axis (Art. 239).

EXAMPLES.

1. Find the centre of percussion of a circular plate of radius a capable of rotating about an axis which touches it.

Here $k_1^2 = \frac{a^2}{4}$, and $h = a$. Hence from (2) we have

$$l = a + \frac{a}{4} = \tfrac{5}{4}a.$$

2. A cylinder is capable of rotating about the diameter of one of its circular ends; find the centre of percussion. Let $a =$ its length, and $b =$ the radius of its base.

Ans. $l = \dfrac{3b^2 + 4a^2}{6a}$.

Hence if $3b^2 = 2a^2$, the centre of percussion will be at the end of the cylinder. If b is very small compared with a, $l = \frac{2}{3}a$; thus if a straight rod of small transverse section is held by one end in the hand, l gives the point at which it may be struck so that the hand will receive no jar.

241. The Principal Radius of Gyration Determined Practically.—Mount the body upon an axis not passing through the centre of gravity, and cause it to oscillate; from the number of oscillations performed in a given time, say an hour, the time of one oscillation is known. Then to find h, which is the distance from the axis to the centre of gravity, attach a spring balance to the lower end, and bring the centre of gravity to a horizontal plane through the axis, which position will be indicated by the maximum reading of the balance. Knowing the maximum reading, R, of the balance, the weight, W, of the body, and the distance, a, from the axis of suspension to the point of attachment, we have from the principle of moments, $Ra = Wh$, from which h is found. Substituting in (3) of Art. 237, this value of h, and for T the time of an oscillation, k_1 becomes known.

242. The Ballistic Pendulum.—An interesting application of the principles of the compound pendulum is the old way of determining the velocity of a bullet or cannon-ball. It is a matter of considerable importance in the Theory of Gunnery to determine the velocity of a bullet as it issues from the mouth of a gun. It was to determine this initial velocity that Mr. Robins about 1743 invented the *Ballistic Pendulum.* This consists of a large thick heavy mass of wood, suspended from a horizontal axis in the shape of a knife-edge, after the manner of a compound pendulum. The gun is so placed that a ball projected from it horizontally strikes this pendulum at rest at a certain point, and gives it a certain angular velocity about its axis. The velocity of the ball is itself too great to be measured directly, but the angular velocity communicated to the pendulum may be made as small as we please by increasing its bulk. The arc of oscillation being measured, the velocity of the bullet can be found by calculation.

The time, which the bullet takes to penetrate, is so short that we may suppose it completed before the pendulum has sensibly moved from its initial position.

Let M be the mass of the pendulum and ball; m that of the ball; v the velocity of the ball at the instant of impact; h the distance of the centre of gravity of the pendulum and ball from the axis of suspension; a the distance of the point of impact from the axis of suspension; ω the angular velocity due to the blow of the ball, and k the radius of gyration of the pendulum and ball. Then since the initial velocity of the bullet is v, its impulse is measured by mv, and therefore from (3) of Art. 236 we have for the initial angular velocity generated in the pendulum by this impulse,

$$\omega = \frac{mva}{Mk^2}; \tag{1}$$

and from (1) of Art. 237 we have for the subsequent motion

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{k^2}\sin\theta. \tag{2}$$

Integrating, and observing that, if α be the angle through which the pendulum moves, we have $\frac{d\theta}{dt} = \omega$ when $\theta = 0$, and $\frac{d\theta}{dt} = 0$ when $\theta = \alpha$, (2) becomes

$$\omega^2 = \frac{2gh}{k^2}(1 - \cos\alpha). \tag{3}$$

Eliminating ω between (1) and (3) we have

$$v = \frac{2Mk}{ma}\sqrt{gh}\,\sin\frac{\alpha}{2}, \tag{4}$$

from which v becomes known, since all the quantities in the second member may be observed, or are known.

We may determine α as follows: At a point in the pendulum at a distance h from the axis of suspension, attach the end of a tape, and let the rest of the tape be wound tightly round a reel; as the pendulum ascends, let a length c be unwound from the reel; then c is the chord of the angle α to the radius h, so that $c = 2h\sin\frac{\alpha}{2}$, which in (4) gives

$$v = \frac{Mkc}{ma}\sqrt{\frac{g}{h}}. \tag{5}$$

The values of k and h may be determined as in Art. 241. If the mouth of the gun is placed near to the pendulum, the value of v, given by (5), must be nearly the velocity of projection.

The velocity may also be determined in the following manner: Let the gun be attached to a heavy pendulum; when the gun is discharged the recoil causes the pendulum to turn round its axis and to oscillate through an arc which can be measured; and the velocity of the bullet can be deduced from the magnitude of this arc. (See Price's Anal. Mech's, Vol. II, p. 231.)

Before the invention of the ballistic pendulum by Mr. Robins in 1743, but little progress had been made in the true theory of military projectiles. Robins' *New Principles of Gunnery* was soon translated into several languages, and Euler added to his translation of it into German an extensive commentary; the work of Euler's being again translated into English in 1784. The experiments of Robins were all conducted with musket balls of about an ounce weight, but they were afterwards continued during several years by Dr. Hutton, who used cannon-balls of from one to nearly three pounds in weight. Hutton used to suspend his cannon as a pendulum, and measure the angle through which it was raised by the discharge. His experiments are still regarded as some of the most trustworthy on smooth-bore guns. See Routh's Rigid Dynamics, p. 94, also Encyclopædia Britannica, Art. Gunnery.

243. Motion of a Heavy Body about a Horizontal Axle through its Centre.—Let the body be a sphere whose radius is R, and weight W, and let a weight P be attached to a cord wound round the circumference of a wheel on the same axle, the radius of the wheel being r; required the distance passed over by P in t seconds.

From (4) of Art. 236 we have

$$\frac{d^2\theta}{dt^2} = \frac{Prg}{Wk_1^2 + Pr^2}.$$

Multiplying by dt and integrating twice, we have

$$\theta = \frac{Prgt^2}{2(Wk_1^2 + Pr^2)}, \tag{1}$$

the constants being zero in both integrations, since the body starts from rest when $t = 0$. The space will be $r\theta$.

EXAMPLES.

1. Let the body be a sphere whose radius is 3 ft. and weight 500 lbs.; let P be 50 lbs., and the radius of the wheel 6 ins.; required the time in which the weight P will descend through 50 ft. (Take $g = 32$.)

Ans. 21 seconds.

2. Let the body be a sphere whose radius is 14 ins. and weight 800 lbs.; let it be moved by a weight of 200 lbs. attached to a cord wound round a wheel the radius of which is one foot; find the number of revolutions of the sphere in eight seconds. (Take $g = 32$.) *Ans.* 51.3.

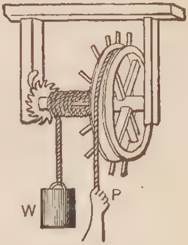

Fig. 97

244. Motion of a Wheel and Axle when a Given Weight *P* Raises a Given Weight *W*.—Let the weights P and W be attached to cords wound round the wheel and axle, respectively, (Fig. 97); let R and r be be the radii of the wheel and axle, and w and w' their weights; required the angular distance passed over in t seconds.

From (4) of Art. 236, we have

$$\frac{d^2\theta}{dt^2} = \frac{PR - Wr}{PR^2 + Wr^2 + \frac{1}{2}wR^2 + \frac{1}{2}w'r^2} g; \qquad (1)$$

$$\therefore \quad \theta = \frac{(PR - Wr)\,t^2}{PR^2 + Wr^2 + \frac{1}{2}wR^2 + \frac{1}{2}w'r^2} \tfrac{1}{2}g. \qquad (2)$$

EXAMPLE.

Let the weight $P = 30$ lbs., $W = 80$ lbs., $w = 8$ lbs., and $w' = 4$ lbs.; and let R and r be 10 ins. and 4 ins.;

required (1) the space passed over by P in 12 seconds if it starts from rest, and (2) the tensions T and T' of the cords, supporting P and W. (Take $g = 32$.)

Ans. (1) 97.79 ft.; (2) $T = 31.28$ lbs.; $T' = 78.64$ lbs.

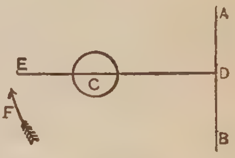

Fig. 98

245. Motion of a Rigid Body about a Vertical Axis.—Let AB be a vertical axis about which the body C, on the horizontal arm ED, revolves, under the action of a constant horizontal force F, applied at the extremity E, perpendicular to ED. Let M be the mass of the body whose centre is C, and r and h the distances ED and CD, respectively. Then from (4) of Art. 236, we have

$$\frac{d^2\theta}{dt^2} = \frac{Fr}{M(k_1^2 + h^2)}.$$

Multiplying by dt and integrating twice, observing that the constants of both integrations are zero, we have

$$\theta = \frac{Frt^2}{2M(k_1^2 + h^2)} \qquad (1)$$

which is the angular space passed over in t seconds.

EXAMPLE.

Let the body be a sphere whose radius is 2 ft., whose weight is 600 lbs., and the distance of whose centre from the axis is 8 ft., and let F be a force of 40 lbs. acting at the end of an arm 10 ft. long; find (1) the number of revolutions which the body will make about the axis in 10 minutes, and (2) the time of one revolution. (Take $g = 32$.)

Ans. (1) 9316.3; (2) 6.2 secs.

$\alpha = 0.325$ radians/sec²

246. Body Rolling down an Inclined Plane.—*A homogeneous sphere rolls directly down a rough inclined plane under the action of gravity. Find the motion.*

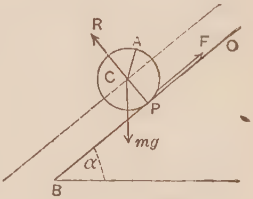

Fig. 99

Let Fig. 99 represent a section of the sphere and plane made by a vertical plane passing through C, the centre of the sphere. Let α be the inclination of the plane to the horizon, a the radius of the sphere, O the point of the plane which was initially touched by the sphere at the point A, P the point of contact at the time t, ACP $= \theta$, which is the angle turned through by the sphere, $m =$ the mass of the sphere, $F =$ the friction acting upwards, $R =$ the pressure of the sphere on the plane. Then it is convenient to choose O for origin and OB for the axis of x; hence OP $= x$.

The forces which act on the sphere are (1) the reaction, R, perpendicular to OB at P, (2) the friction, F, acting at P along PO, and (3) its weight, mg, acting vertically at C the centre. Now C evidently moves along a straight line parallel to the plane; so that for its motion of translation we have, by resolving along the plane,

$$m\frac{d^2x}{dt^2} = mg \sin \alpha - F. \qquad (1)$$

The sphere evidently rotates about its point of contact with the plane; but it may be considered as rotating at any instant about its centre C as fixed; and the angular velocity of C at that instant in reference to P is the same as that of P in reference to C. From (4) of Art. 236, we have for the motion of rotation

$$mk_1^2\frac{d^2\theta}{dt^2} = Fa \qquad (2)$$

and since the plane is perfectly rough, so that the sphere does not slide, we have

$$x = a\theta. \tag{3}$$

Multiplying (1) by a and adding the result to (2), we get

$$ma\frac{d^2x}{dt^2} + mk_1{}^2\frac{d^2\theta}{dt^2} = mag \sin \alpha. \tag{4}$$

Differentiating (3) twice we get $\frac{d^2x}{dt^2} = a\frac{d^2\theta}{dt^2}$, which united to (4) gives

$$\frac{d^2x}{dt^2} = \frac{a^2}{a^2 + k_1{}^2} g \sin \alpha. \tag{5}$$

Since the sphere is homogeneous, $k_1{}^2 = \frac{2}{5}a^2$, and (5) becomes

$$\frac{d^2x}{dt^2} = \frac{5}{7}g \sin \alpha \tag{6}$$

which gives the acceleration down the plane.

If the sphere had been sliding down a *smooth* plane, the acceleration would have been $g \sin \alpha$ (Art. 144); so that two-sevenths of gravity is used in turning the sphere, and five-sevenths in urging the sphere down the plane.

Integrating (6) twice, and supposing the sphere to start from rest, we have

$$x = \tfrac{5}{14}g \cdot \sin \alpha \cdot t^2$$

which gives the space passed over in the time t.

Resolving perpendicular to the plane, we have

$$R = mg \cos \alpha.$$

Cor.—If the rolling body were a circular cylinder with its axis horizontal, then $k_1{}^2 = \frac{1}{2}a^2$, and (5) becomes

$$\frac{d^2x}{dt^2} = \frac{2}{3}g \sin \alpha; \tag{7}$$

so that one-third of gravity is used in turning the cylinder, and two-thirds in urging it down the plane.

From (7) we have

$$x = \tfrac{1}{3}g \sin \alpha \cdot t^2 \qquad (8)$$

which gives the space passed over in the time t from rest.

247. Motion of a Falling Body under the Action of an Impulsive Force not Directly through its Centre.—*A string is wound round the circumference of a reel, and the free end is attached to a fixed point. The reel is then lifted up and let fall so that at the moment when the string becomes tight it is vertical, and tangent to the reel. The whole motion being supposed to take place in one plane, determine the effect of the impulse.*

The reel at first will fall vertically without rotation. Let v be the velocity of the centre at the moment when the string becomes tight; v', ω the velocity of the centre and the angular velocity just after the impulse; T the impulsive tension; m the mass of the reel, and a its radius.

Just after the impact the part of the reel in contact with the string has no velocity, and at this instant the reel rotates about this part; but it may be considered as rotating about its axis as fixed, and the angular velocity of its axis, at this instant, in reference to the part in contact is the same as that of the latter in reference to the former. The impulsive tension is

$$T = m\,(v - v'). \qquad (1)$$

Hence from (3) of Art. 236, we have for the motion of rotation

$$mk_1^2\omega = m\,(v - v')\,a. \qquad (2)$$

Since the part of the reel in contact with the string has no velocity at the instant of impact, we have

$$v' = a\omega. \tag{3}$$

Solving (2) and (3) we have

$$\omega = \frac{av}{a^2 + k_1{}^2}. \tag{4}$$

If the reel be a homogeneous cylinder, $k_1{}^2 = \frac{a^2}{2}$, and we have from (3) and (4)

$$\omega = \tfrac{2}{3}\frac{v}{a}, \quad v' = \tfrac{2}{3}v, \tag{5}$$

and from (1) we have for the impulsive tension,

$$T = \tfrac{1}{3}mv.$$

Cor.—*To find the subsequent motion.* The centre of the reel *begins* to descend vertically; and as there is no horizontal force on it, it will continue to descend in a vertical straight line, and throughout all its subsequent motion the string will be vertical. The motion may therefore be easily investigated, as in Art. 246, since it is similar to the case of a body rolling down an inclined plane which is vertical, the tension of the string taking the place of the friction along the plane. Hence putting $\alpha = \frac{\pi}{2}$, and letting the friction $F =$ the finite tension of the string, we have, from (1) and (7) of Art. 246,

$$F = \tfrac{1}{3}mg, \quad \text{and} \quad \frac{d^2x}{dt^2} = \tfrac{2}{3}g;$$

that is, the finite tension of the string is one-third of the weight, and the reel descends with a uniform acceleration of $\frac{2}{3}g$.

Since the initial velocity of the reel from (5) is $\frac{2}{3}v$, we have, for the space descended in the time t after the impact, from (8) of Art. 246,

$$x = \tfrac{2}{3}vt + \tfrac{1}{3}gt^2. \quad \text{(See Routh's Rigid Dynamics, p. 131.)}$$

EXAMPLES.

1. A thin rod of steel 10 ft. long, oscillates about an axis passing through one end of it; find (1) the time of an oscillation, and (2) the number of oscillations it makes in a day. *Ans.* (1) 1.434 sec.; (2) 60254.

2. A pendulum oscillates about an axis passing through its end; it consists of a steel rod 60 ins. long, with a rectangular section $\frac{1}{2}$ by $\frac{1}{4}$ of an inch; on this rod is a steel cylinder 2 in. in diameter and 4 in. long; when the ends of the rod and cylinder are set square, find the time of an oscillation. *Ans.* 1.174 secs.

3. Determine the radius of gyration with reference to the axis of suspension of a body that makes 73 oscillations in 2 minutes, the distance of the centre of gravity from the axis being 3 ft. 2 in. *Ans.* 5.267 ft.

4. Determine the distance between the centres of suspension and oscillation of a body that oscillates in $2\frac{1}{2}$ sec. *Ans.* 20.264 ft.

5. A thin circular plate oscillates about an axis passing through the circumference; find the length of the simple equivalent pendulum, (1) when the axis touches the circle and is in its plane, and (2) when it is at right angles to the plane of the circle. *Ans.* (1) $\frac{5}{4}a$; (2) $\frac{3}{2}a$.

6. A cube oscillates about one of its edges; find the length of the simple equivalent pendulum, the edge being $= 2a$. *Ans.* $\frac{4}{3}a\sqrt{2}$.

7. A prism, whose cross section is a square, each side being $= 2a$, and whose length is l, oscillates about one of its upper edges; find the length of the simple equivalent pendulum. *Ans.* $\frac{2}{3}\sqrt{4a^2 + l^2}$.

8. An elliptic lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum passes through the centre of oscillation; prove that the eccentricity is $\frac{1}{2}$.

9. The density of a rod varies as the distance from one end; find the axis perpendicular to it about which the time of oscillation is a minimum, l being the length of the rod.

Ans. The distance of the axis from the centre of gravity is $\frac{l}{6}\sqrt{2}$.

10. Find the axis about which an elliptic lamina must oscillate that the time of oscillation may be a minimum.

Ans. The axis must be parallel to the major axis, and bisect the semi-minor axis.

11. Find the centre of percussion of a cube which rotates about an axis parallel to the four parallel edges of the cube, and equidistant from the two nearer, as well as from the two farther edges. Let $2a$ be a side of the cube, and let c be the distance of the rotation-axis from its centre of gravity.

Ans. $l = c + \frac{2a^2}{3c}$, where l is the distance from the rotation-axis to the centre of percussion.

12. Find the centre of percussion of a sphere which rotates about an axis tangent to its surface.

Ans. $l = \frac{7}{5}a$.

13. Let the body in Art. 243, be a sphere whose radius is 17 ins. and weight 1200 lbs.; let it be moved by a weight of 250 lbs. attached to a cord wound round a wheel whose

radius is 15 ins.; find the number of revolutions of the sphere in 10 seconds. $(g = 32.)$ *Ans.* 58.77.

14. Let the body in Art. 243 be a sphere of radius 8 ins. and weight 500 lbs.; let it be moved by a weight of 100 lbs. attached to a cord wound round a wheel whose radius is 6 in.; find the number of revolutions of the sphere in 5 seconds. $(g = 32\frac{1}{6}.)$ *Ans.* 28.09.

15. In Art. 244, let the weight $P = 40$ lbs., $W = 100$ lbs., $w = 12$ lbs., and $w' = 6$ lbs.; and let R and r be 12 ins. and 7 ins.; required (1) the space passed over by P in 16 secs. if it starts from rest, and (2) the tensions T and T' of the cords supporting P and W. $(g = 32)$.
Ans. (1) 926.5; (2) $T = 49.04$ lbs., $T' = 86.81$ lbs.

16. In Art. 244, let the weight $P = 25$ lbs., $W = 60$ lbs., $w = 6$ lbs., and $w' = 2$ lbs.; and let R and r be 8 in. and 3 in.; required (1) the space passed over by P in 10 secs. if it starts from rest, and (2) the tensions T and T' of the cords supporting P and W. $(g = 32\frac{1}{6}.)$
Ans. (1) 109.92 ft.; (2) $T = 23.29$ lbs.; $T' = 61.54$ lbs.

✓ $\alpha = 0.34$/

17. In Art. 245, let the body be a sphere whose radius is 3 ft., whose weight is 800 lbs., and the distance of whose centre from the axis is 9 ft.; and let F be a force of 60 lbs. acting at the end of an arm 12 ft. long; find (1) the number of revolutions which the body will make about the axis in 12 min., and (2) the time of one revolution. $(g = 32.)$ *Ans.* (1) 14043.6; (2) 6.07 secs.

18. In Ex. 17, let the radius = one foot, the weight = 100 lbs., the distance of centre from axis = 5 ft., and $F = 25$ lbs. acting at end of arm 8 ft. long; find (1) the number of revolutions which the body will make about the axis in 5 min., and (2) the time of one revolution. $(g = 32\frac{1}{6}.)$ *Ans.* (1) 18139.09; (2) 2.23 secs.

19. If the body in Art. 247 be a homogeneous sphere, the string being round the circumference of a great circle,

find (1) the angular velocity just after the impulse, and (2) the impulsive tension. *Ans.* $\frac{5v}{7a}$; (2) $\frac{2}{7}mv$.

20. A bar, l feet long, falls vertically, retaining its horizontal position till it strikes a fixed obstacle at one-quarter the length of the bar from the centre; find (1) the angular velocity of the bar, (2) the linear velocity of its centre just after the impulse, and (3) the impulsive force, the velocity at the instant of the impulse being v.

Ans. (1) $\frac{12v}{7l}$; (2) $\frac{3}{7}v$; (3) $\frac{4}{7}mv$.

21. A bar, 40 ft. long, falls through a vertical height of 50 ft., retaining its horizontal position till one end strikes a fixed obstacle 60 ft. above the ground; find (1) its angular velocity, (2) the linear velocity of its centre just after the impulse; (3) the number of revolutions it will make before reaching the ground, (4) the whole time of falling to the ground, and (5) its linear velocity on reaching the ground.

Ans. (1) 2.12; (2) 42.43; (3) 0.345; (4) 2.79; (5) 75.10.

CHAPTER VIII.

MOTION OF A SYSTEM OF RIGID BODIES IN SPACE

248. The Equations of Motion of a System of Rigid Bodies obtained by D'Alembert's Principle.— Let (x, y, z) be the position of the particle m at the time t referred to any set of rectangular axes fixed in space, and $X, Y, Z,$ the axial components of the impressed accelerating forces acting on the same particle. Then $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2},$ are the axial components of the accelerations of the particle; and by D'Alembert's Principle (Art. 235) the forces,

$$m\left(X-\frac{d^2x}{dt^2}\right), \quad m\left(Y-\frac{d^2y}{dt^2}\right), \quad m\left(Z-\frac{d^2z}{dt^2}\right),$$

acting on m together with similar forces acting on every particle of the system, are in equilibrium. Hence by the principles of Statics (Art. 65) we have the following six equations of motion:

$$\left.\begin{aligned} \Sigma m\left(X-\frac{d^2x}{dt^2}\right) &= 0, \\ \Sigma m\left(Y-\frac{d^2y}{dt^2}\right) &= 0, \\ \Sigma m\left(Z-\frac{d^2z}{dt^2}\right) &= 0. \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} \Sigma m(yZ-zY) - \Sigma m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right) &= 0, \\ \Sigma m(zX-xZ) - \Sigma m\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right) &= 0, \\ \Sigma m(xY-yX) - \Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right) &= 0. \end{aligned}\right\} \tag{2}$$

By means of these six equations the motion of a rigid body acted on by any finite forces, may be determined. They lead immediately to two important propositions, one of which enables us to calculate the motion of *translation* of the body in space; and the other the motion of *rotation*.

249. Independence of the Motion of Translation of the Centre of Gravity, and of Rotation about an Axis Passing through it.—Let $(\bar{x}, \bar{y}, \bar{z})$ be the position of the centre of gravity of the body at the time t, referred to fixed axes, (x, y, z) the position of the particle m referred to the same axes, (x', y', z') the position of m referred to a system of axes passing through the centre of gravity and parallel to the fixed axes, and M the whole mass. Then

1. $$x = \bar{x} + x', \quad y = \bar{y} + y', \quad z = \bar{z} + z'. \tag{1}$$

Since the origin of the movable system is at the centre of gravity, we have (Art. 59)

$$\Sigma mx' = \Sigma my' = \Sigma mz' = 0; \tag{2}$$

$$\therefore \quad \Sigma m \frac{d^2x'}{dt^2} = \Sigma m \frac{d^2y'}{dt^2} = \Sigma m \frac{d^2z'}{dt^2} = 0. \tag{3}$$

Also $\Sigma mx = M\bar{x}, \quad \Sigma my = M\bar{y}, \Sigma mz = M\bar{z};$

$$\therefore \quad \Sigma m \frac{d^2x}{dt^2} = M\frac{d^2\bar{x}}{dt^2}, \quad \Sigma m \frac{d^2y}{dt^2} = M\frac{d^2\bar{y}}{dt^2}, \quad \Sigma m \frac{d^2z}{dt^2} = M\frac{d^2\bar{z}}{dt^2}.$$

Substituting these values in (1) of Art. 248, we have

$$\left.\begin{aligned} M\frac{d^2\bar{x}}{dt^2} &= \Sigma \cdot mX; \\ M\frac{d^2\bar{y}}{dt^2} &= \Sigma \cdot mY; \\ M\frac{d^2\bar{z}}{dt^2} &= \Sigma \cdot mZ. \end{aligned}\right\} \tag{4}$$

These three equations do not contain the co-ordinates of the point of application of the forces, and are the same as those which would be obtained for the motion of the centre of gravity supposing the forces all applied at that point. Hence

The motion of the centre of gravity of a system acted on by any forces is the same as if all the mass were collected at the centre of gravity and all the forces were applied at that point parallel to their former directions.

2. Differentiating (1) twice we have

$$\frac{d^2x}{dt^2} = \frac{d^2\bar{x}}{dt^2} + \frac{d^2x'}{dt^2}, \quad \frac{d^2y}{dt^2} = \frac{d^2\bar{y}}{dt^2} + \frac{d^2y'}{dt^2},$$

$$\frac{d^2z}{dt^2} = \frac{d^2\bar{z}}{dt^2} + \frac{d^2z'}{dt^2}.$$

Substituting these values in the first of equations (2) of Art. 248, we have

$$\Sigma m\,[(\bar{y} + y')\,Z - (\bar{z} + z')\,Y]$$

$$- \Sigma m\left[(\bar{y} + y')\left(\frac{d^2\bar{z}}{dt^2} + \frac{d^2z'}{dt^2}\right) - (\bar{z} + z')\left(\frac{d^2\bar{y}}{dt^2} + \frac{d^2y'}{dt^2}\right)\right] = 0.$$

Performing the operations indicated we get

$$\bar{y}\Sigma\cdot m\left(Z - \frac{d^2\bar{z}}{dt^2}\right) - \bar{y}\Sigma\cdot m\,\frac{d^2z'}{dt^2} + \Sigma my'\left(Z - \frac{d^2z'}{dt^2}\right)$$

$$- \frac{d^2\bar{z}}{dt^2}\,\Sigma my' - \bar{z}\Sigma m\left(Y - \frac{d^2\bar{y}}{dt^2}\right) + \bar{z}\Sigma m\,\frac{d^2y'}{dt^2}$$

$$- \Sigma mz'\left(Y - \frac{d^2y'}{dt^2}\right) + \frac{d^2\bar{y}}{dt^2}\,\Sigma mz' = 0.$$

Omitting the 1st, 2d, 4th, 5th, 6th, and 8th terms which vanish by reason of (2), (3), and (4), we have

$$\Sigma m \left(y' \frac{d^2 z'}{dt^2} - z' \frac{d^2 y'}{dt^2} \right) = \Sigma m \left(y'Z - z'Y \right),$$

similarly from the other two equations of (2) we have

$$\left.\begin{aligned} \Sigma m \left(z' \frac{d^2 x'}{dt^2} - x' \frac{d^2 z'}{dt^2} \right) &= \Sigma m \left(z'X - x'Z \right), \\ \Sigma m \left(x' \frac{d^2 y'}{dt^2} - y' \frac{d^2 x'}{dt^2} \right) &= \Sigma m \left(x'Y - y'X \right). \end{aligned}\right\} \quad (5)$$

These three equations do not contain the co-ordinates of the centre of gravity, and are exactly the equations we would have obtained if we had regarded the centre of gravity as a fixed point, and taken it as the origin of moments. Hence

The motion of a body, acted on by any forces, about its centre of gravity is the same as if the centre of gravity were fixed and the same forces acted on the body. That is, from (4) *the motion of translation of the centre of gravity of the body is independent of its rotation;* and from (5) *the rotation of the body is independent of the translation of its centre.*

These two important propositions are called respectively, *the principles of the conservation of the motions of translation and rotation.*

Sch.—By the first principle the problem of finding the motion of the centre of gravity of a system, however complex the system may be, is reduced to the problem of finding the motion of a single particle. By the second principle the problem of finding the angular motion of a free body in space is reduced to that of determining the motion of that body about a fixed point.

REM.—In using the first principle it should be noticed that the impressed forces are to be applied at the centre of gravity *parallel* to their former directions. Thus, if a rigid body be moving under the influence of a central force, the motion of the centre of gravity is *not* generally the same as if the whole mass were collected at the centre of gravity and it were *then* acted on by the same central force. What the principle asserts is, that, if the attraction of the central force on each element of the body be found, the motion of the centre of gravity is the same as if *these* forces were applied at the centre of gravity parallel to their original directions.

250. The Principle of the Conservation of the Centre of Gravity.—Suppose that a material system is acted on by no other forces than the mutual attractions of its parts; then the impressed accelerating forces are zero, which give

$$\Sigma X = \Sigma Y = \Sigma Z = 0;$$

therefore from (4) of Art. 249, we get

$$\frac{d^2\bar{x}}{dt^2} = 0, \quad \frac{d^2\bar{y}}{dt^2} = 0, \quad \frac{d^2\bar{z}}{dt^2} = 0;$$

$$\therefore \quad \frac{d\bar{x}}{dt} = v_0 \cos \alpha, \quad \frac{d\bar{y}}{dt} = v_0 \cos \beta, \quad \frac{d\bar{z}}{dt} = v_0 \cos \gamma. \quad (1)$$

where v_0 is the velocity of the centre of gravity when $t = 0$, and α, β, γ, are the angles which its direction makes with the axes. Therefore, calling v the velocity of the centre of gravity at the time t, we have

$$v = \sqrt{\frac{d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2}{dt^2}} = v_0, \quad (2)$$

which is evidently constant.

If $v_0 = 0$, *the centre of gravity remains at rest.*

Integrating (1) we get

$$\bar{x} = v_0 t \cos\alpha + a, \quad \bar{y} = v_0 t \cos\beta + b,$$

$$\bar{z} = v_0 t \cos\gamma + c;$$

$$\therefore \quad \frac{\bar{x} - a}{\cos\alpha} = \frac{\bar{y} - b}{\cos\beta} = \frac{\bar{z} - c}{\cos\gamma} \qquad (3)$$

(a, b, c) being the place of the centre of gravity of the system when $t = 0$. As (3) are the equations of a straight line it follows that *the motion of the centre of gravity is rectilinear.*

Hence when a material system is in motion under the action of forces, none of which are external to the system, then the centre of gravity moves uniformly in a straight line or remains at rest.

Rem.—Thus the motion of the centre of gravity of a system of particles is not altered by their mutual collision, whatever degree of elasticity they may have, because a reaction always exists equal and opposite to the action. If an explosion occurs in a moving body, whereby it is broken into pieces, the line of motion and the velocity of the centre of gravity of the body are not changed by the explosion; thus the motion of the centre of gravity of the earth is unaltered by earthquakes; volcanic explosions on the moon will not change its motion in space. The motion of the centre of gravity of the solar system is not affected by the mutual and reciprocal action of its several members; it is changed only by the action of forces external to the system.

251. The Principle of the Conservation of Areas.—If x, y be the rectangular, and r, θ the polar co-ordinates of a particle, we have

$$x\frac{dy}{dt} - y\frac{dx}{dt} = x^2\frac{d}{dt}\left(\frac{y}{x}\right)$$

$$= r^2\cos^2\theta\frac{d}{dt}(\tan\theta) = r^2\frac{d\theta}{dt}. \qquad (1)$$

Now $\frac{1}{2}r^2 d\theta$ is the elementary area described round the origin in the time dt by the projection of the radius vector of the particle on the plane of xy, (Art. 182.) If twice this polar area be multiplied by the mass of the particle, it is called *the area conserved* by the particle in the time dt round the axis of z. Hence

$$\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

is called the *area conserved* by the system.

Let dA_x, dA_y, dA_z be twice the areas described by the projections of the radius vector of the particle m on the planes of yz, zx, xy, respectively; then from (1) we have

$$\Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \Sigma m\frac{dA_z}{dt};$$

and differentiating we get

$$\Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = \Sigma m\frac{d^2A_z}{dt^2}. \qquad (2)$$

If the impressed accelerating forces are zero the first member of (2) is zero, from (5) of Art. 249; therefore the second member is zero. Hence

$$\Sigma m\frac{d^2A_z}{dt^2} = 0;$$

similarly $$\Sigma m \frac{d^2 A_x}{dt^2} = 0, \quad \Sigma m \frac{d^2 A_y}{dt^2} = 0;$$

and therefore by integration

$$\Sigma m \frac{dA_x}{dt} = h, \quad \Sigma m \frac{dA_y}{dt} = h', \quad \Sigma m \frac{dA_z}{dt} = h'',$$

h, h', h'' being constants.

$$\therefore \quad \Sigma m A_x = ht, \quad \Sigma m A_y = h't, \quad \Sigma m A_z = h''t;$$

the limits of integration being such that the areas and the time begin simultaneously. Thus, the sum of the products of the mass of every particle, and the projection of the area described by its radius vector on each co-ordinate plane, varies as the time. This theorem is called *the principle of the conservation of areas.* That is,

When a material system is in motion under the action of forces, none of which are external to the system, then the sum of the products of the mass of each particle by the projection, on any plane, of the area described by the radius vector of this particle measured from any fixed point, varies as the time of motion.

252. Conservation of Vis Viva or Energy.*—Let (x, y, z) be the place of the particle m at the time t, and let X, Y, Z be the axial components of the impressed accelerating forces acting on the particle, as in Art. 248. The axial components of the effective forces acting on the same particle at any time t are

$$m \frac{d^2 x}{dt^2}, \quad m \frac{d^2 y}{dt^2}, \quad m \frac{d^2 z}{dt^2}.$$

If the effective forces on all the particles be reversed, they will be in equilibrium with the whole group of impressed forces (Art. 235). Hence, by the principle of virtual velocities (Art. 104), we have

---

* See Art. 189.

$$\Sigma m\left[\left(X-\frac{d^2x}{dt^2}\right)\delta x+\left(Y-\frac{d^2y}{dt^2}\right)\delta y+\left(Z-\frac{d^2z}{dt^2}\right)\delta z\right]=0, \quad (1)$$

where δx, δy, δz are any small arbitrary displacements of the particle m parallel to the axes, consistent with the connection of the parts of the system with one another at the time t.

Now the spaces actually described by the particle m during the instant after the time t parallel to the axes are consistent with the connection of the parts of the system with each other, and hence we may take the arbitrary displacements, δx, δy, δz, to be respectively equal to the actual displacements, $\frac{dx}{dt}\delta t$, $\frac{dy}{dt}\delta t$, $\frac{dz}{dt}\delta t$, of the particle.*

Making this substitution, (1) becomes

$$\Sigma m\left(\frac{d^2x}{dt^2}\frac{dx}{dt}+\frac{d^2y}{dt^2}\frac{dy}{dt}+\frac{d^2z}{dt^2}\frac{dz}{dt}\right)$$

$$=\Sigma m\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right).$$

Integrating, we get

$$\Sigma mv^2-\Sigma mv_0^2=2\Sigma m\int(Xdx+Ydy+Zdz), \quad (2)$$

where v and v_0 are the velocities of the particle m at the times t and t_0.

The first member of (2) is twice the vis viva or kinetic energy of the system acquired in its motion from the time t_0 to the time t, under the action of the given forces.

---

* That is, although δx is not equal to dx, yet the *ratio* of δx to dx is equal to the ratio of δt to dt.

The second member expresses twice the work done by these forces in the same time (Art. 189).

If the second member of (2) be an exact differential of a function of x, y, z, so that it equals $df(x, y, z)$; then taking the definite integral between the limits x, y, z and x_0, y_0, z_0, corresponding to t and t_0, (2) becomes

$$\Sigma mv^2 - \Sigma mv_0^2 = 2f(x, y, z) - 2f(x_0, y_0, z_0). \qquad (3)$$

Now the second member of (2) is an exact differential so far as any particle m is acted on by a central force whose centre is fixed at (a, b, c), and which is a function of the distance r between the centre and (x, y, z) the place of m. Thus, let P be the central force $= f(r)$, say: then

$$X = \frac{x-a}{r} f(r), \quad Y = \frac{y-b}{r} f(r) \quad Z = \frac{z-c}{r} f(r);$$

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2;$$

$$\therefore \quad rdr = (x-a)\,dx + (y-b)\,dy + (z-c)\,dz;$$

$$\therefore \quad m\,(Xdx + Ydy + Zdz) = mf(r)\,dr;$$

which is an exact differential; substituting this in the second member of (2), it

$$= 2m\int_{r_0}^{r} f(r)\,dr,$$

where the limits r and r_0 correspond to t and t_0.

Also, the second member of (2) is an exact differential, so far as any two particles of the system are attracted towards or repelled from each other by a force which varies as the mass of each, and is a function of the distance between them. Let m and m' be any two particles; let (x, y, z), (x', y', z') be their places at the time t; r their distance apart; $P = f(r)$, the mutual action of the unit mass of each particle. Then the whole attractive force of

m on m' is Pm, and the whole attractive force of m' on m is Pm'; and we have

$$X = m\frac{x - x'}{r}P, \quad Y = m\frac{y - y'}{r}P, \quad Z = m\frac{z - z'}{r}P;$$

$$X' = -m\frac{x - x'}{r}P, \quad Y' = -m\frac{y - y'}{r}P, \quad Z' = -m\frac{z - z'}{r}P.$$

Also $\quad r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2.$

Therefore for these two particles, we have

$$\begin{aligned} &m\,(Xdx + Ydy + Zdz) + m'\,(X'dx' + Y'dy' + Z'dz') \\ &= \frac{mm'}{r}P\,[(x - x')\,(dx - dx') + (y - y')\,(dy - dy') \\ &\qquad\qquad + (z - z')\,(dz - dz')] \\ &\qquad\qquad = mm' f(r)\,dr; \end{aligned}$$

which is an exact differential. The same reasoning applied to every two particles in the system must lead to a similar result; so that finally the second member of (2)

$$= 2mm'\int_{r_0}^{r} f(r)\,dr,$$

where the limits r and r_0 correspond to t and t_0, so that the integral will be a function solely of the initial and final co-ordinates of the particles of the system.

Hence, *when a material system is in motion under the action of forces, none of which are external to the system, then the change of the vis viva of the system, in passing from one position to another, depends only on the two positions of the system, and is independent of the path described by each particle of the system.*

This theorem is called *the principle of the conservation of vis viva or energy.*

COR. 1.—If a system be under the action of no external forces, we have $X = Y = Z = 0$, and hence the vis viva of the system is constant.

COR. 2.—Let gravity be the only force acting on the system. Let the axis of z be vertical and positive downwards, then we have $X = 0$, $Y = 0$, $Z = g$. Hence (2) becomes

$$\Sigma mv^2 - \Sigma mv_0^2 = 2\Sigma m\,(z - z_0).$$

But if $\bar{z}$ and $\bar{z}_0$ are the distances from the plane of xy to the centre of gravity of the system at the times t and t_0, and if M is the mass of the system, we have

$$M\bar{z} = \Sigma mz, \quad M\bar{z}_0 = \Sigma mz_0;$$

$$\therefore\ \Sigma mv^2 - \Sigma mv_0^2 = 2Mg\,(\bar{z} - \bar{z}_0). \tag{4}$$

That is, *the increase of vis viva of the system depends only on the vertical distance over which the centre of gravity passes; and therefore the vis viva is the same whenever the centre of gravity passes through a given horizontal plane.*

REM.—The principle of vis viva was first used by Huyghens in his determination of the centre of oscillation of a body (Art. 237, Rem.).

The advantage of this principle is that it gives at once a relation between the velocities of the bodies considered and the co-ordinates which determine their positions in space, so that when, from the nature of the problem, the position of all the bodies may be made to depend on one variable, the equation of vis viva is sufficient to determine the motion.

Suppose a weight mg to be placed at any height h above the surface of the earth. As it falls through a height z, the force of gravity does work which is measured by mgz. The weight has acquired a velocity v, and therefore its vis viva is $\frac{1}{2}mv^2$ which is equal to mgz (Art. 217). If the weight falls through the remainder of the height h, gravity does more work which is measured by $mg\,(h - z)$. When the weight has reached the ground, it has fallen as far as the circumstances of the case permit, and gravity has done work which is measured by mgh, and can do no more work until the weight has been lifted up again. Hence, throughout the motion when the weight has descended through any space z, its vis viva, $\frac{1}{2}mv^2 (= mgz)$, together with the work that can be done during the rest of the descent, $mg(h - z)$, is constant and equal to mgh, the work done by gravity during the whole descent h.

If we complicate the motion by making the weight work some machine during its descent, the same theorem is still true. The vis viva of the weight, when it has descended any space z, is equal to the work mgz which has been done by gravity during this descent, diminished by the work done on the machine. Hence, as before, the vis viva together with the difference between the work done by gravity and that done on the machine during the remainder of the descent is constant and equal to the excess, of the work done by gravity over that done on the machine during the whole descent. (See Routh's Rigid Dynamics, p. 270.)

253. Composition of Rotations.—It is often necessary to compound rotations about axes which meet at a point. When a body is said to have angular velocities about three different axes at the same time, it is only meant that the motion may be determined as follows: Divide the whole time into a number of infinitesimal intervals each equal to dt. During each of these, turn the body round the three axes successively, through angles $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$. The result will be the same in whatever order the rotations take place. The final displacement of the body is the diagonal of the parallelopiped described on these three lines as sides, and is therefore independent of the order of the rotations. Since then the three successive rotations are quite independent, they may be said to take place simultaneously.

Hence we infer that angular velocities and angular accelerations may be compounded and resolved by the same rules and in the same way as if they were linear. Thus, an angular velocity ω about any given axis may be resolved into two, $\omega \cos \alpha$ and $\omega \sin \alpha$, about axes at right angles to each other and making angles α and $\frac{\pi}{2} - \alpha$ with the given axis.

Also, if a body have angular velocities ω_1, ω_2, ω_3 about three axes at right angles, they are together equivalent to a single angular velocity ω, where $\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$, about an axis inclined to the given axes at angles whose cosines are respectively $\frac{\omega_1}{\omega}, \frac{\omega_2}{\omega}, \frac{\omega_3}{\omega}$.

254. Motion of a Rigid Body referred to Fixed Axes.—Let us suppose that one point in the body is fixed. Let this point be taken as the origin of co-ordinates, and let the axes OX, OY, OZ be any directions fixed in space and at right angles to one another. The body at the time t is turning about some axis of instantaneous rotation (Art. 240). Let its angular velocity about this axis be ω, and let this be resolved into the angular velocities ω_1, ω_2, ω_3 about the co-ordinate axes. It is required to find the resolved linear velocities, $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, parallel to the axes of co-ordinates, of a particle m at the point P, (x, y, z), in terms of the angular velocities about the axes.

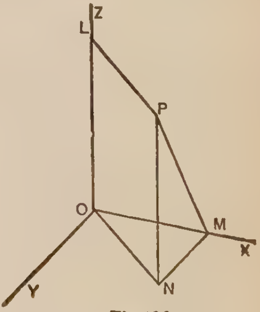

Fig. 100

These angular velocities are supposed positive when they tend the same way round the axes that positive couples tend in Statics (Art. 65). Thus the positive directions of ω_1, ω_2, ω_3 are respectively from y to z about x, from z to x about y, and from x to y about z; and those negative which act in the opposite directions.

Let us determine the velocity of P parallel to the axis of z. Let PN be the ordinate z, and draw PM perpendicular to the axis of x. The velocity of P due to rotation about OX is $\omega_1 PM$. Resolving this parallel to the axes of y and z, and reckoning those linear velocities positive which tend *from* the origin, and *vice versa*, we have the velocity

along $\quad MN = -\omega_1 PM \cos NPM = -\omega_1 z$;

and along $\quad NP = \omega_1 PM \sin NPM = \omega_1 y$.

Similarly the velocity due to the rotation about OY parallel to OX is $\omega_2 z$, and parallel to OZ is $-\omega_2 x$. And that due to the rotation about OZ parallel to OX is $-\omega_3 y$, and parallel to OY is $\omega_3 x$.

Adding together those velocities which are parallel to the same axes, we have for the velocities of P parallel to the axes of x, y, and z, respectively,

$$\left.\begin{aligned} \frac{dx}{dt} &= \omega_2 z - \omega_3 y, \\ \frac{dy}{dt} &= \omega_3 x - \omega_1 z, \\ \frac{dz}{dt} &= \omega_1 y - \omega_2 x. \end{aligned}\right\} \tag{1}$$

255. Axis of Instantaneous Rotation.—Every particle in the axis of instantaneous rotation is at rest relative to the origin; hence, for these particles each of the first members of (1) in Art. 254, will reduce to zero, and we have

$$\left.\begin{aligned} \omega_2 z - \omega_3 y &= 0, \\ \omega_3 x - \omega_1 z &= 0, \\ \omega_1 y - \omega_2 x &= 0, \end{aligned}\right\} \tag{1}$$

which are the equations of the axis of instantaneous rotation, the third equation being a necessary consequence of the first two; hence,

$$x = \frac{\omega_1}{\omega_3} z, \quad y = \frac{\omega_2}{\omega_3} z; \tag{2}$$

that is, the instantaneous axis is a straight line passing through the origin which is at rest at the instant considered; and the whole body must, for the instant, rotate about this line.

COR.—Denote by α, β, γ the angles which this axis makes with the co-ordinate axes x, y, z, respectively, then (Anal. Geom., Art. 175) we have

$$\cos \alpha = \frac{\omega_1}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}};$$

$$\cos \beta = \frac{\omega_2}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}};$$

$$\cos \gamma = \frac{\omega_3}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}};$$

which gives the position of the instantaneous axis in terms of the angular velocities about the co-ordinate axes.

256. The Angular Velocity of the Body about the Axis of Instantaneous Rotation.—The angular velocity of the body about this axis will be the same as that of any single particle chosen at pleasure. Let the particle be taken on the axis of x; if from it we draw a perpendicular, p, to the instantaneous axis, then the distance of the particle from the origin being x, we have

$$p = x \sin \alpha = x\sqrt{1 - \cos^2 \alpha} = x\sqrt{\frac{\omega_2^2 + \omega_3^2}{\omega_1^2 + \omega_2^2 + \omega_3^2}}.$$

Since, for this particle, $y = 0$, $z = 0$, we have from (1) of Art. 254, for the absolute velocity,

$$V = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt} = x\sqrt{\omega_2^2 + \omega_3^2},$$

and hence, for the angular velocity v, we have

$$v = \frac{V}{p} = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2},$$

which is the angular velocity required.

257. Euler's Equations.—*To determine the general equations of motion of a body about a fixed point.*

Let the fixed point O be taken as origin; let (x, y, z) be the place of any particle m, at the time t, referred to any rectangular axes fixed in space, and let Ox_1, Oy_1, Oz_1 be the principal axes of the body (Art. 231). Differentiating (1) of Art. 254 with respect to t, we have

$$\frac{d^2x}{dt^2} = z\frac{d\omega_2}{dt} - y\frac{d\omega_3}{dt} + \omega_2(\omega_1 y - \omega_2 x) - \omega_3(\omega_3 x - \omega_1 z),$$

$$\frac{d^2y}{dt^2} = x\frac{d\omega_3}{dt} - z\frac{d\omega_1}{dt} + \omega_3(\omega_2 z - \omega_3 y) - \omega_1(\omega_1 y - \omega_2 x),$$

$$\frac{d^2z}{dt^2} = y\frac{d\omega_1}{dt} - x\frac{d\omega_2}{dt} + \omega_1(\omega_3 x - \omega_1 z) - \omega_2(\omega_2 z - \omega_3 y).$$

Denoting by L, M, N, the first terms respectively of (2), (Art. 248), and substituting the above values of $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ in the last of these equations, we get

$$\left.\begin{aligned}\Sigma m\,(x^2+y^2)\frac{d\omega_3}{dt}-\Sigma myz\cdot\frac{d\omega_2}{dt}-\Sigma mxz\frac{d\omega_1}{dt}\\ -\Sigma mxy\,(\omega_1{}^2-\omega_2{}^2)+\Sigma m\,(x^2-y^2)\,\omega_1\omega_2\\ -\Sigma myz\omega_1\omega_3+\Sigma mxz\omega_2\omega_3\end{aligned}\right\}=N. \quad (1)$$

The other two equations may be treated in the same way.

The coefficients in this equation are the moments and products of inertia of the body with regard to axes fixed in space (Art. 224), and are therefore variable as the body moves about. Let ω_x, ω_y, ω_z be the angular velocities about the principal axes. Since the axes fixed in space are perfectly arbitrary, let them be so chosen that the principal axes are coinciding with them at the moment under consideration. Then *at this moment* we have (Art. 232),

$$\Sigma mxy = 0, \quad \Sigma myz = 0, \quad \Sigma mzx = 0;$$

also $\omega_1 = \omega_x$, $\omega_2 = \omega_y$, $\omega_3 = \omega_z$; and likewise $\dfrac{d\omega_1}{dt} = \dfrac{d\omega_x}{dt}$, etc.* Hence, denoting by A, B, C, the moments of inertia about the principal axes (Art. 231), (1) becomes

$$C\frac{d\omega_3}{dt} - (A - B)\,\omega_x\omega_y = N,$$

in which all the coefficients are constants; and similarly for the other two equations.

Hence, uniting them in order, and retaining the letters ω_1, ω_2, ω_3, since they are equal to ω_x, ω_y, ω_z, the three equations of motion of the body referred to the principal axes at the fixed point are

---

* $\dfrac{d\omega_1}{dt} = \dfrac{d\omega_x}{dt}$, for the changes in the two angular velocities, ω_1 and ω_x, during a given small time after the axis of x_1 coincides with the axis of x, will differ only by a quantity which depends upon the angle passed through by the axis of x_1 during that given small time; the difference between ω_1 and ω_x will therefore be an infinitesimal of the second order and therefore their derivatives will be equal. (See Pratt's Mech., p. 428. For further demonstration of this equality, the student is referred to Routh's Rigid Dynamics, pp. 188 and 189.)

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 &= L, \\ B\frac{d\omega_2}{dt} - (C - A)\,\omega_3\omega_1 &= M, \\ C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 &= N, \end{aligned}\right\} \qquad (2)$$

These are called Euler's Equations.

SCH.—If the body is moving so there is no point in it which is fixed in space, the motion of the body about its centre of gravity is the same as if that point were fixed.

It is clear that, instead of referring the motion of the body to the principal axes at the fixed point, as Euler has done, we may use any axes fixed in the body. But these are in general so complicated as to be nearly useless.

258. Motion of a Body about a Principal Axis through its Centre of Gravity.—*If a body rotate about one of its principal axes passing through the centre of gravity, this axis will suffer no pressure from the centrifugal force.*

Let the body rotate about the axis of z; then if ω be its angular velocity, the centrifugal force of any particle m will be (Art. 198, Cor. 1)

$$m\omega^2\rho = m\omega^2\sqrt{x^2 + y^2},$$

which gives for the x- and y-components $m\omega^2 x$ and $m\omega^2 y$; and the moments of these forces with respect to the axes of y and x are for the whole body

$$\Sigma m\omega^2 xz, \quad \text{and} \quad \Sigma m\omega^2 yz.$$

But these are each equal to zero when the axis of rotation is a principal axis (Art. 232); hence, the centrifugal force will have no tendency to *incline* the axis of z towards the plane of xy. In this case the only effect of the forces $m\omega^2x$ and $m\omega^2y$ on the axis is to move it parallel to itself, or to translate the body in the directions of x and y. But the *sum* of all these forces is

$$\Sigma m\omega^2x \quad \text{and} \quad \Sigma m\omega^2y,$$

each of which is equal to zero when the axis of rotation passes through the centre of gravity; hence we conclude that, *when a body rotates about one of its principal axes passing through its centre of gravity, the rotation causes no pressure upon the axis.*

If the body rotates about this axis it will continue to rotate about it if the axis be removed. On this account a principal axis through the centre of gravity is called *an axis of permanent rotation.**

Sch.—If the body be free, and it begins to rotate about an axis very near to a principal axis, the centrifugal force will cause the axis of rotation to change continually, inasmuch as the foregoing conditions cannot obtain, and this axis of rotation will either continually oscillate about the principal axis, always remaining very near to it, or else it will remove itself indefinitely from the principal axis. Hence, whenever we observe a free body rotating about an axis during any time, however short, we may infer that it has continued to rotate about that axis from the beginning of the motion, and that it will continue to rotate about it for ever, unless checked by some extraneous obstacle. (See Young's Mechs., p. 230, also Venturoli, pp. 135 and 160.)

---

* Pratt's Mechs., p. 422. Called also *a natural axis of rotation*, see Young's Mechs., p. 230; also *an invariable axis*, see Price's Mechs., Vol. II, p. 257.

259. Velocity about a Principal Axis when there are no Accelerating Forces.—In this case $L = M = N = 0$ in (2) of Art. 257; also A, B, C are constant for the same body; and if we put

$$\frac{B-C}{A} = F, \quad \frac{C-A}{B} = G, \quad \frac{A-B}{C} = H,$$

(2) of Art. 257 becomes

$$d\omega_1 = F\omega_2\omega_3 dt, \quad d\omega_2 = G\omega_3\omega_1 dt,$$
$$d\omega_3 = H\omega_1\omega_2 dt.$$

Put $\omega_1\omega_2\omega_3 dt = d\phi$, and we have (1)

$$\omega_1 d\omega_1 = F d\phi, \quad \omega_2 d\omega_2 = G d\phi, \quad \omega_3 d\omega_3 = H d\phi;$$

and integrating, we get

$$\omega_1^2 = 2F\phi + a^2, \quad \omega_2^2 = 2G\phi + b^2, \quad \omega_3^2 = 2H\phi + c^2. \quad (2)$$

where a, b, c are the initial values of ω_1, ω_2, ω_3; hence from (1) and (2)

$$dt = \frac{d\phi}{\sqrt{(2F\phi + a^2)(2G\phi + b^2)(2H\phi + c^2)}}. \quad (3)$$

Suppose now the body begins to turn about only one of the principal axes, say the axis of x, with the angular velocity a, then $b = 0$, $c = 0$, and (3) becomes

$$dt = \frac{1}{2\sqrt{GH}} \cdot \frac{d\phi}{\phi\sqrt{2F\phi + a^2}}.$$

Replacing $2F\phi + a^2$ by its value ω_1^2, and $d\phi$ by its value $\frac{\omega_1 d\omega_1}{F}$, we have

$$dt = \frac{1}{\sqrt{GH}} \cdot \frac{d\omega_1}{\omega_1^2 - a^2};$$

and integrating, we get

$$C + t\sqrt{GH} = \frac{1}{2a}\log\frac{\omega_1 - a}{\omega_1 + a};$$

$$\therefore \quad e^{2aC} \,.\, e^{2at\sqrt{GH}} = \frac{\omega_1 - a}{\omega_1 + a}. \tag{4}$$

The constant C must be determined so that when $t = 0$, ω_1 is the initial velocity a; hence $e^{2aC} = 0$ or $C = -\infty$, which makes the first member of (4) zero for every value of t. Hence, at any time t, we must have $\omega_1 = a$; and therefore from (2) $\phi = 0$, and $\omega_2 = \omega_3 = 0$. *Consequently the impressed velocity about one of the principal axes of rotation continues perpetual and uniform*, as before shown (Art. 258).

260. The Integral of Euler's Equations.—*A body revolves about its centre of gravity acted on by no forces but such as pass through that point; to integrate the equations of motion.*

As the only forces acting on the body are those which pass through its centre of gravity, they create no moment of rotation about an axis passing through that centre; and therefore (2) of Art. 257 become

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 &= 0,\\ B\frac{d\omega_2}{dt} - (C - A)\,\omega_3\omega_1 &= 0,\\ C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 &= 0, \end{aligned}\right\} \tag{1}$$

the principal axes being drawn through the centre of gravity.

Multiply these equations severally (1) by ω_1, ω_2, ω_3; and (2) by $A\omega_1$, $B\omega_2$, $C\omega_3$, and add; then we have

$$\left.\begin{aligned} A\omega_1\frac{d\omega_1}{dt}+B\omega_2\frac{d\omega_2}{dt}+C\omega_3\frac{d\omega_3}{dt}=0;\\ A^2\omega_1\frac{d\omega_1}{dt}+B^2\omega_2\frac{d\omega_2}{dt}+C^2\omega_3\frac{d\omega_3}{dt}=0;\end{aligned}\right\} \qquad (2)$$

integrating, we have

$$\left.\begin{aligned} A\omega_1^2+B\omega_2^2+C\omega_3^2=h^2;\\ A^2\omega_1^2+B^2\omega_2^2+C^2\omega_3^2=k^2;\end{aligned}\right\} \qquad (3)$$

where h^2 and k^2 are the constants of integration.

Eliminating ω_3^2 from (3), we have

$$A(A-C)\,\omega_1^2+B(B-C)\,\omega_2^2=k^2-Ch^2;$$

$$\therefore\quad \omega_2^2=\frac{1}{B(B-C)}\left[k^2-Ch^2-A(A-C)\,\omega_1^2\right]; \qquad (4)$$

$$\text{and}\quad \omega_3^2=\frac{1}{C(C-B)}\left[k^2-Bh^2-A(A-B)\,\omega_1^2\right]. \qquad (5)$$

Substituting these values of ω_2 and ω_3 in the first of equations (1), we have

$$\frac{d\omega_1}{dt}+\left[\frac{(A-C)(A-B)}{BC}\left(\omega_1^2-\frac{k^2-Ch^2}{A(A-C)}\right)\left(\frac{k^2-Bh^2}{A(A-B)}-\omega_1^2\right)\right]^{\frac{1}{2}}=0 \qquad (6)$$

which is generally an elliptic transcendent, and so does not admit of integration in finite terms. In certain particular cases it may be integrated, which will give the value of ω_1 in terms of t, and if this value be substituted in (4) and (5), the values of ω_2 and ω_3 in terms of t will be known, and thus, in these cases, the problem admits of complete solution.

COR.—Let ω_x, ω_y, ω_z be the axial components of the initial angular velocity about the principal axes when $t = 0$; then integrating the first of (2), and taking the limits corresponding to t and 0, we have

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = A\omega_x^2 + B\omega_y^2 + C\omega_z^2. \quad (7)$$

Let α, β, γ be the direction-angles of the instantaneous axis at the time t relative to the principal axes; so that, if ω is the instantaneous angular velocity, and Σmr^2 is the moment of inertia relative to that axis, we have (Art. 253). $\omega_1 = \omega \cos \alpha$, $\omega_2 = \omega \cos \beta$, $\omega_3 = \omega \cos \gamma$, which substituted in (7), gives

$$\begin{aligned} A\omega_x^2 + B\omega_y^2 + C\omega_z^2 &= \omega^2 (A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma) \\ &= \omega^2 \Sigma mr^2 \text{ (Art. 232, Cor.)} \\ &= \Sigma mv^2 \\ &= \text{the vis viva of the body;} \end{aligned}$$

from which it appears that *the vis viva of the body is constant throughout the whole motion.*

REM.—An application of the general equations of rotatory motion (Art. 257), which is of great interest and importance, is that of the rotatory phenomena of the earth under the action of the attracting forces of the sun and the moon, the rotation being considered relative to the centre of gravity and an axis passing through it, just as if the centre of gravity was a fixed point (Art. 249, Sch.); and the problem treated as purely a mathematical one. Also, in addition to the sun and the moon, the problem may be extended so as to include the action of all the other bodies whose influence affects the motion of the earth's rotation. In fact the investigation of the motion of a system of bodies in space might be continued at great length; but such investigations would be clearly beyond the limits proposed in this treatise. The student who desires to continue this interesting subject, is referred to more extended works.*

EXAMPLES.

1. A hollow spherical shell is filled with fluid, and rolls down a rough inclined plane; determine its motion.

Let M and M' be the masses of the shell and fluid respectively, k and k' their radii of gyration respectively about a diameter, and a and a' the radii of the exterior and interior surfaces of the shell; then ~~using~~ the same notation as in Art. 246, we have

$$(M + M')\frac{d^2x}{dt^2} = (M + M')\,g\sin\alpha - F. \qquad (1)$$

As the spherical shell rotates in its descent down the plane, the fluid has only motion of translation; so that the equation of rotation is

$$Mk^2\frac{d^2\theta}{dt^2} = Fa. \qquad (2)$$

Multiplying (1) by a^2 and (2) by a, and adding, we have

$$[(M + M')\,a^2 + Mk^2]\,\frac{d^2x}{dt^2} = (M + M')\,a^2g\sin\alpha. \quad (3)$$

If the interior were solid, and rigidly joined to the shell, the equation of motion would be

---

* **See Price's Mech's, Vol. II, Pratt's Mech's, Routh's Rigid Dynamics, La Place's Mécanique Céleste, etc.**

$$[(M+M')\,a^2+Mk^2+M'k'^2]\,\frac{d^2x}{dt^2}=(M+M')\,a^2g\sin\alpha. \quad (4)$$

Integrating (3) and (4) twice, and denoting by s and s' the spaces through which the centre moves during the time t in these two cases respectively, we have

$$\frac{s}{s'}=\frac{(M+M')\,a^2+Mk^2+M'k'^2}{(M+M')\,a^2+Mk^2} \quad (5)$$

so that a greater space is described by the sphere which has the fluid than by that which has the solid in its interior.

If the densities of the solid and the fluid are the same, we have from (5), by Art. 233, Ex. 14,

$$\frac{s}{s'}=\frac{7a^5}{7a^5-2a'^5}. \quad \text{(Price's Anal. Mechs., Vol. II, p. 368).}$$

2. A homogeneous sphere rolls down within a rough spherical bowl; it is required to determine the motion.

Let a be the radius of the sphere, and b the radius of the bowl; and let us suppose the sphere to be placed in the bowl at rest. Let $OCQ=\phi$, $QPA=\theta$, $BCO=\alpha$, $\omega=$ the angular velocity of the ball about an axis through its centre P, $k=$ the corresponding radius of gyration; $OM=x$, $MP=y$; $m=$ the mass of the ball. Then

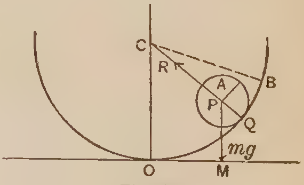

Fig. 101

$$m\frac{d^2x}{dt^2}=-R\sin\phi+F\cos\phi; \quad (1)$$

$$m\frac{d^2y}{dt^2}=R\cos\phi+F\sin\phi-mg; \quad (2)$$

$$mk^2 \frac{d\omega}{dt} = aF. \tag{3}$$

Also $x = (b - a) \sin \phi$; $y = b - (b - a) \cos \phi$.

$$\therefore \quad \frac{d^2x}{dt^2} = (b - a) \cos \phi \frac{d^2\phi}{dt^2} - (b - a) \sin \phi \left(\frac{d\phi}{dt}\right)^2, \tag{4}$$

$$\frac{d^2y}{dt^2} = (b - a) \sin \phi \frac{d^2\phi}{dt^2} + (b - a) \cos \phi \left(\frac{d\phi}{dt}\right)^2, \tag{5}$$

$$\frac{d^2x}{dt^2} \cos \phi + \frac{d^2y}{dt^2} \sin \phi = (b - a) \frac{d^2\phi}{dt^2}. \tag{6}$$

$$\therefore \quad m (b - a) \frac{d^2\phi}{dt^2} = F - mg \sin \phi. \tag{7}$$

Now to determine the angular velocity of the ball, we must estimate the angle described by a fixed line in it, as PA, from a line fixed in direction, as PM, and the ratio of the infinitesimal increase of this angle to that of the time will be the angular velocity of the ball.

$$\therefore \quad \omega = \frac{dMPA}{dt} = \frac{d\phi}{dt} + \frac{d\theta}{dt}.$$

Since the sphere does not slide, $a\theta = b (\alpha - \phi)$;

$$\therefore \quad \omega = \frac{a - b}{a} \frac{d\phi}{dt};$$

$$\therefore \quad \frac{d\omega}{dt} = \frac{a - b}{a} \frac{d^2\phi}{dt^2}; \tag{8}$$

from (3), (7), and (8) we get

$$(b - a) \frac{d^2\phi}{dt^2} = -\tfrac{5}{7} g \sin \phi; \tag{9}$$

$$\therefore \quad (b-a)\left(\frac{d\phi}{dt}\right)^2 = \frac{10g}{7}(\cos\phi - \cos\alpha). \qquad (10)$$

Substituting (9) in (7) we have

$$F = \tfrac{2}{7}mg\sin\phi. \qquad (11)$$

Substituting (4), (9), (10), (11) in (1) we have

$$R = \frac{mg}{7}(17\cos\phi - 10\cos\alpha);$$

therefore the pressure at the lowest point

$$= \frac{mg}{7}(17 - 10\cos\alpha);$$

and the pressure of the ball on the bowl vanishes when

$$\cos\phi = \tfrac{10}{17}\cos\alpha.$$

Cor.—If the ball rolls over a small arc at the lowest part of the bowl, so that α and ϕ are always small, $\cos\alpha$, and $\cos\phi$ may be replaced by $1 - \frac{\alpha^2}{2}$ and $1 - \frac{\phi^2}{2}$ respectively; and from (10) we have

$$\frac{-d\phi}{(\alpha^2 - \phi^2)^{\frac{1}{2}}} = \left[\frac{5g}{7(b-a)}\right]^{\frac{1}{2}} dt;$$

$$\therefore \quad \phi = \alpha\cos\left[\frac{5g}{7(b-a)}\right]^{\frac{1}{2}} t;$$

thus the ball comes to rest at points whose angular distance is α on both sides of 0, the lowest point of the bowl; and the periodic time is

$$\pi\left[\frac{7(b-a)}{5g}\right]^{\frac{1}{2}};$$

therefore the oscillations are performed in the same time as those of a simple pendulum whose length is $\frac{7}{5}(b-a)$, (Art. 194). (Price's Anal. Mech's, Vol. II, p. 369.)

3. A homogeneous sphere has an angular velocity ω about its diameter, and gradually contracts, remaining constantly homogeneous, till it has half the original diameter; required the final angular velocity. *Ans.* 4ω.

4. If the earth were a homogeneous sphere, at what point must it be struck, that it may receive its present velocity of translation and of rotation, the former being 68000 miles per hour nearly? *Ans.* 24 miles nearly from the centre.

5. A homogeneous sphere rolls down a rough inclined plane; the inclined plane rests on a smooth horizontal plane, along which it slides by reason of the pressure of the sphere; required the motions of the inclined plane and of the centre of the sphere.

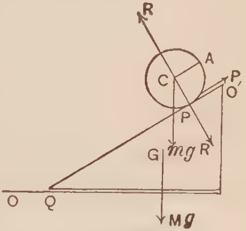

Fig. 102

Let m = the mass of the sphere, M = the mass of the inclined plane, a = the radius of the sphere, α = the angle of the inclined plane, Q its apex; O the place of Q when $t=0$; O' the point on the plane which was in contact with the point A of the sphere when $t=0$, at which time we may suppose all to be at rest; $ACP=\theta$, the angle through which the sphere has revolved in the time t.

Let O be the origin, and let the horizontal and vertical lines through it be the axes of x and y; $OQ=x'$; and let (x, y) (h, k) be the places of the centre of the sphere at the times $t=t$ and $t=0$ respectively. Then the equations of motion of the sphere are

$$m\frac{d^2x}{dt^2}=F\cos\alpha-R\sin\alpha,$$

$$m\frac{d^2y}{dt^2} = F\sin\alpha + R\cos\alpha - mg,$$

$$\tfrac{2}{5}ma^2\frac{d^2\theta}{dt^2} = aF;$$

and the equation of motion of the plane is

$$M\frac{d^2x'}{dt^2} = -F\cos\alpha + R\sin\alpha.$$

From the geometry we have

$$x = h + x' - a\theta\cos\alpha,$$

$$y = k - a\theta\sin\alpha.$$

From these equations we obtain

$$x' = \frac{m\cos\alpha}{m+M}a\theta$$

$$= \frac{5m\sin\alpha\cos\alpha}{7(m+M) - 5m\cos^2\alpha}\cdot\frac{gt^2}{2};$$

$$\therefore\quad x = h - \frac{5M\sin\alpha\cos\alpha}{7(m+M) - 5m\cos^2\alpha}\cdot\frac{gt^2}{2},$$

$$y = k - \frac{5(m+M)\sin^2\alpha}{7(m+M) - 5m\cos^2\alpha}\cdot\frac{gt^2}{2}$$

which give the values of x and y in terms of t.

Also we obtain

$$(m+M)(x-h)\sin\alpha - M(y-k)\cos\alpha = 0;$$

which is the equation of the path described by the centre of the sphere; and therefore this path is a straight line.

6. A heavy solid wheel in the form of a right circular cylinder, is composed of two substances, whose volumes are

equal, and whose densities are ρ and ρ'; these substances are arranged in two different forms; in one case, that whose density is ρ occupies the central part of the wheel, and the other is placed as a ring round it; in the second case, the places of the substances are interchanged; t and t' are the times in which the wheels roll down a given rough inclined plane from rest; show that

$$t^2 : t'^2 :: 5\rho + 7\rho' : 5\rho' + 7\rho.$$

7. A homogeneous sphere moves down a rough inclined plane, whose angle of inclination α to the horizon is greater than that of the angle of friction; it is required to show (1) that the sphere will roll without sliding when μ is equal to or greater than $\frac{2}{7}\tan\alpha$, and (2) that it will slide and roll when μ is less than $\frac{2}{7}\tan\alpha$, where μ is the coefficient of friction.

8. In the last example show that the angular velocity of the sphere at the time t from rest $= \dfrac{5\mu g\cos\alpha}{2a}t$.

9. If the body moving down the plane is a circular cylinder of radius $= a$, with its axis horizontal, show that the body will slide and roll, or roll only, according as α is greater or not greater than $\tan^{-1} 3\mu$.

DATE DUE

Demco, Inc. 38-293

Bowser 50696

WITHDRAWN

3326800 0125961 6

33268000125961 6

QA 805 .B78 1904

BOWSER, EDWARD A. 1845-1910.

AN ELEMENTARY TREATISE ON
ANALYTIC MECHANICS